1 INTRODUCTION

1-1. WHAT IS A GRAPH?

A *linear*† *graph* (or simply a *graph*) $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \ldots\}$ called *vertices*, and another set $E = \{e_1, e_2, \ldots\}$, whose elements are called *edges*, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i, v_j associated with edge e_k are called the *end vertices* of e_k. The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices. Often this diagram itself is referred to as the graph. The object shown in Fig. 1-1, for instance, is a graph.

Observe that this definition permits an edge to be associated with a vertex pair (v_i, v_i). Such an edge having the same vertex as both its end vertices is called a *self-loop* (or simply a *loop*. The word loop, however, has a different meaning in electrical network theory; we shall therefore use the term self-loop to avoid confusion). Edge e_1 in Fig. 1-1 is a self-loop. Also note that

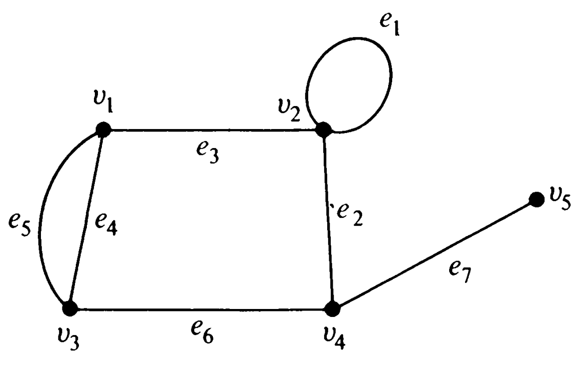

Fig. 1-1 Graph with five vertices and seven edges.

†The adjective "linear" is dropped as redundant in our discussions, because by a graph we always mean a linear graph. There is no such thing as a nonlinear graph.

the definition allows more than one edge associated with a given pair of vertices, for example, edges e_4 and e_5 in Fig. 1-1. Such edges are referred to as *parallel edges.*

A graph that has neither self-loops nor parallel edges is called a *simple graph.* In some graph-theory literature, a graph is defined to be only a simple graph, but in most engineering applications it is necessary that parallel edges and self-loops be allowed; this is why our definition includes graphs with self-loops and/or parallel edges. Some authors use the term *general graph* to emphasize that parallel edges and self-loops are allowed.

It should also be noted that, in drawing a graph, it is immaterial whether the lines are drawn straight or curved, long or short: what is important is the incidence between the edges and vertices. For example, the two graphs drawn in Figs. 1-2(a) and (b) are the same, because incidence between edges and vertices is the same in both cases.

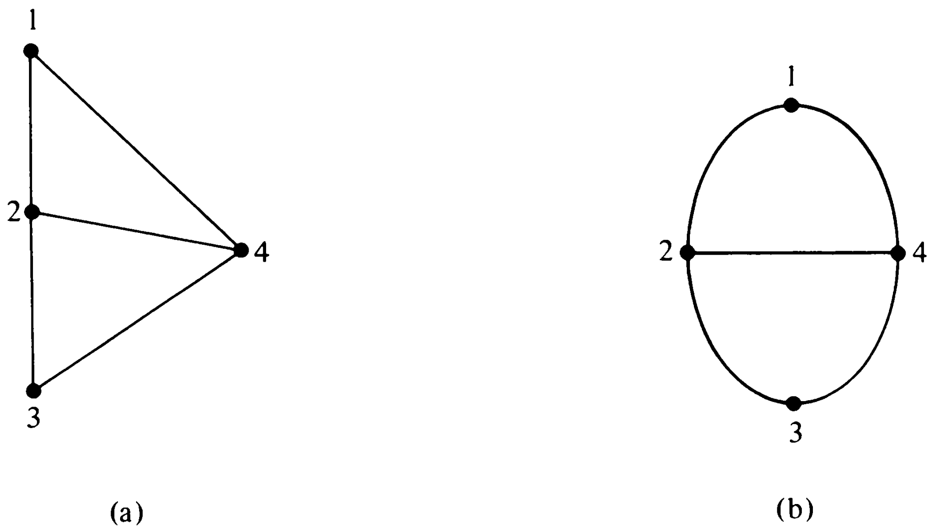

Fig. 1-2 Same graph drawn differently.

In a diagram of a graph, sometimes two edges may seem to intersect at a point that does not represent a vertex, for example, edges e and f in Fig. 1-3. Such edges should be thought of as being in different planes and thus having no common point. (Some authors break one of the two edges at such a crossing to emphasize this fact.)

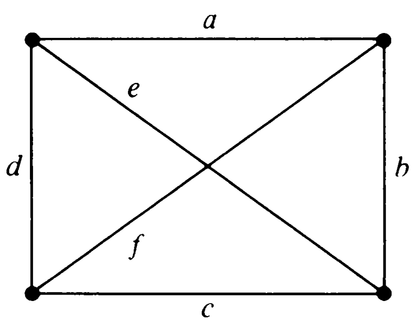

Fig. 1-3 Edges e and f have no common point.

A graph is also called a *linear complex*, a *1-complex*, or a *one-dimensional complex*. A vertex is also referred to as a *node*, a *junction*, a *point*, *0-cell*, or an *0-simplex*. Other terms used for an edge are a *branch*, a *line*, an *element*, a *1-cell*, an *arc*, and a *1-simplex*. In this book we shall generally use the terms graph, vertex, and edge.

1-2. APPLICATIONS OF GRAPHS

Because of its inherent simplicity, graph theory has a very wide range of applications in engineering, in physical, social, and biological sciences, in linguistics, and in numerous other areas. A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them. The following are four examples from among hundreds of such applications.

Königsberg Bridge Problem: The Königsberg bridge problem is perhaps the best-known example in graph theory. It was a long-standing problem until solved by Leonhard Euler (1707–1783) in 1736, by means of a graph. Euler wrote the first paper ever in graph theory and thus became the originator of the theory of graphs as well as of the rest of topology. The problem is depicted in Fig. 1-4.

Two islands, *C* and *D*, formed by the Pregel River in Königsberg (then the capital of East Prussia but now renamed Kaliningrad and in West Soviet Russia) were connected to each other and to the banks *A* and *B* with seven bridges, as shown in Fig. 1-4. The problem was to start at any of the four land areas of the city, *A*, *B*, *C*, or *D*, walk over each of the seven bridges *exactly* once, and return to the starting point (without swimming across the river, of course).

Euler represented this situation by means of a graph, as shown in Fig. 1-5. The vertices represent the land areas and the edges represent the bridges.

As we shall see in Chapter 2, Euler proved that a solution for this problem does not exist.

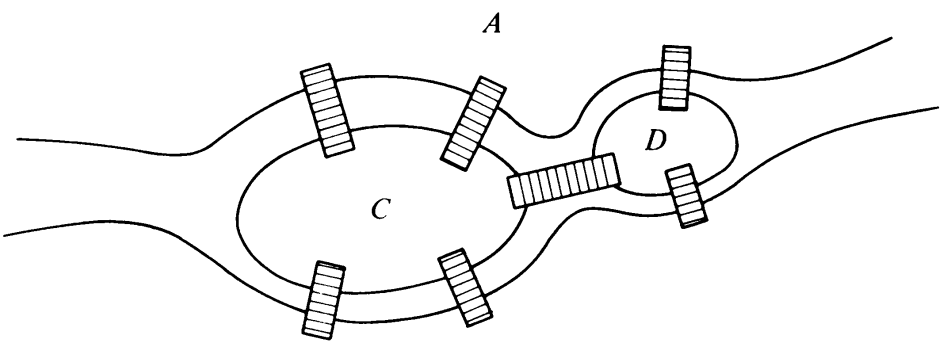

Fig. 1-4 Königsberg bridge problem.

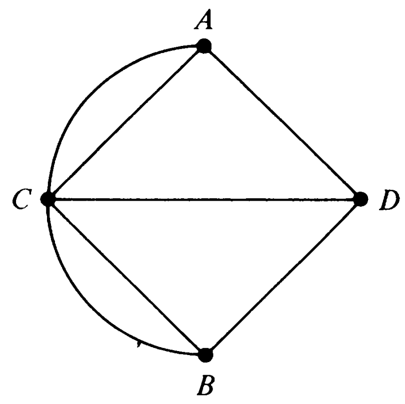

Fig. 1-5 Graph of Königsberg bridge problem.

The Königsberg bridge problem is the same as the problem of drawing figures without lifting the pen from the paper and without retracing a line (Problems 2-1 and 2-2). We all have been confronted with such problems at one time or another.

Utilities Problem: There are three houses (Fig. 1-6) H_1, H_2, and H_3, each to be connected to each of the three utilities—water (W), gas (G), and elec-

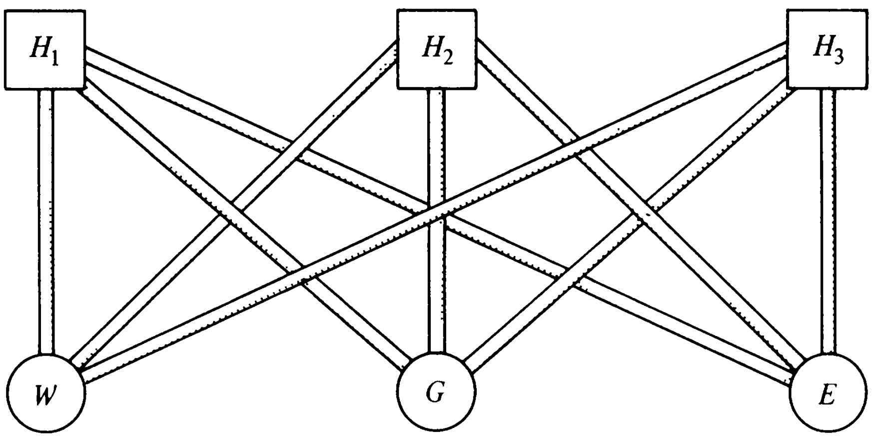

Fig. 1-6 Three-utilities problem.

tricity (E)—by means of conduits. Is it possible to make such connections without any crossovers of the conduits?

Figure 1-7 shows how this problem can be represented by a graph—the conduits are shown as edges while the houses and utility supply centers are vertices. As we shall see in Chapter 5, the graph in Fig. 1-7 cannot be drawn in the plane without edges crossing over. Thus the answer to the problem is no.

Electrical Network Problems: Properties (such as transfer function and input impedance) of an electrical network are functions of only two factors:

1. The nature and value of the elements forming the network, such as resistors, inductors, transistors, and so forth.
2. The way these elements are connected together, that is, the topology of the network.

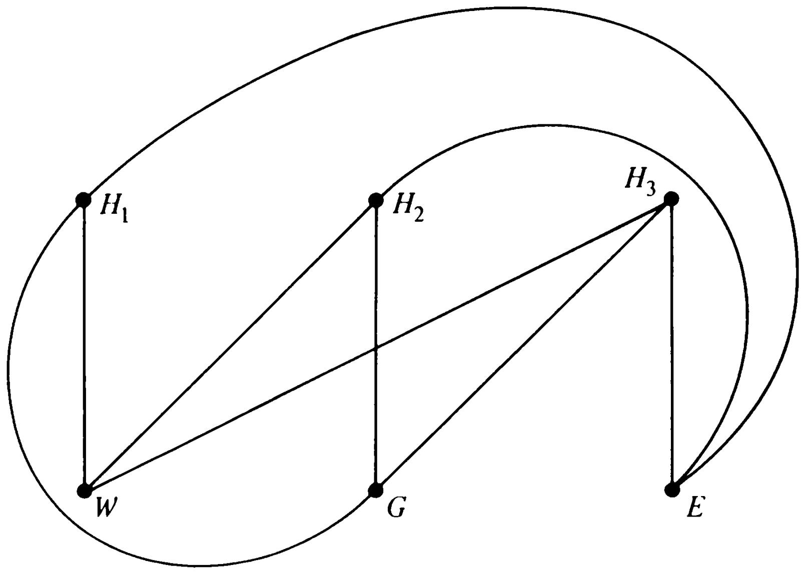

Fig. 1-7 Graph of three-utilities problem.

Since there are only a few different types of electrical elements, the variations in networks are chiefly due to the variations in topology. Thus electrical network analysis and synthesis are mainly the study of network topology. In the topological study of electrical networks, factor 2 is separated from 1 and is studied independently. The advantage of this approach will be clearer in Chapter 13, a chapter devoted solely to applying graph theory to electrical networks.

The topology of a network is studied by means of its graph. In drawing a graph of an electrical network the junctions are represented by vertices, and branches (which consist of electrical elements) are represented by edges, regardless of the nature and size of the electrical elements. An electrical network and its graph are shown in Fig. 1-8.

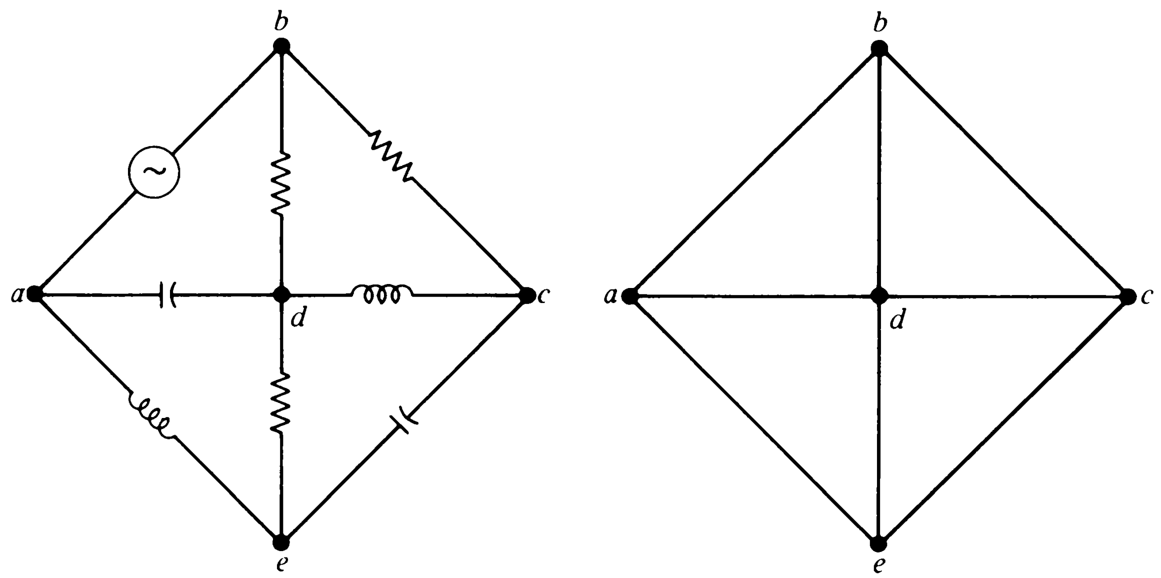

Fig. 1-8 Electrical network and its graph.

Seating Problem: Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbors at each lunch. How many days can this arrangement last?

This situation can be represented by a graph with nine vertices such that each vertex represents a member, and an edge joining two vertices represents the relationship of sitting next to each other. Figure 1-9 shows two possible

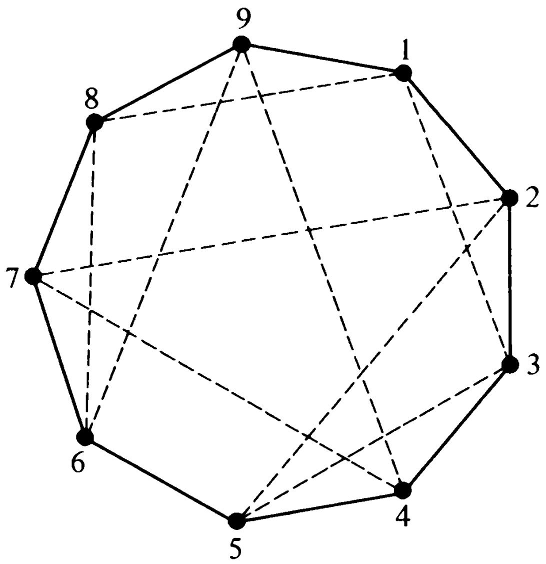

Fig. 1-9 Arrangements at a dinner table.

seating arrangements—these are 1 2 3 4 5 6 7 8 9 1 (solid lines), and 1 3 5 2 7 4 9 6 8 1 (dashed lines). It can be shown by graph-theoretic considerations that there are only two more arrangements possible. They are 1 5 7 3 9 2 8 4 6 1 and 1 7 9 5 8 3 6 2 4 1. In general it can be shown that for n people the number of such possible arrangements is

$$\frac{n-1}{2}, \quad \text{if } n \text{ is odd,}$$

and

$$\frac{n-2}{2}, \quad \text{if } n \text{ is even.}$$

The reader has probably noticed that three of the four examples of applications above are puzzles and not engineering problems. This was done to avoid introducing at this stage background material not pertinent to graph theory. More substantive applications will follow, particularly in the last four chapters.

1-3. FINITE AND INFINITE GRAPHS

Although in our definition of a graph neither the vertex set V nor the edge set E need be finite, in most of the theory and almost all applications these sets are finite. A graph with a finite number of vertices as well as a finite number of edges is called a *finite graph;* otherwise, it is an *infinite graph.* The graphs in Figs. 1-1, 1-2, 1-5, 1-7, and 1-8 are all examples of finite graphs. Portions of two infinite graphs are shown in Fig. 1-10.

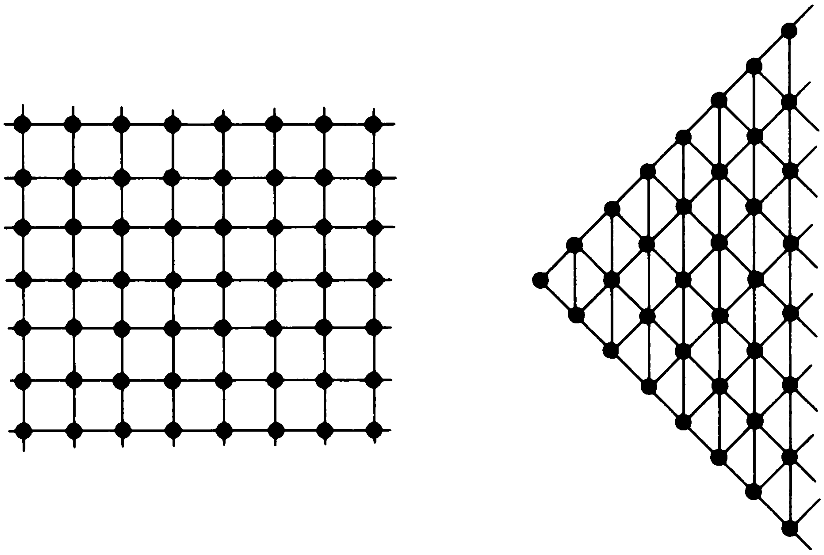

Fig. 1-10 Portions of two infinite graphs.

In this book we shall confine ourselves to the study of finite graphs, and unless otherwise stated the term "graph" will always mean a finite graph.

1-4. INCIDENCE AND DEGREE

When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be *incident with* (*on* or *to*) each other. In Fig. 1-1, for example, edges e_2, e_6, and e_7 are incident with vertex v_4. Two nonparallel edges are said to be *adjacent* if they are incident on a common vertex. For example, e_2 and e_7 in Fig. 1-1 are adjacent. Similarly, two vertices are said to be adjacent if they are the end vertices of the same edge. In Fig. 1-1, v_4 and v_5 are adjacent, but v_1 and v_4 are not.

The number of edges incident on a vertex v_i, with self-loops counted twice, is called the *degree*, $d(v_i)$, of vertex v_i. In Fig. 1-1, for example, $d(v_1) =$

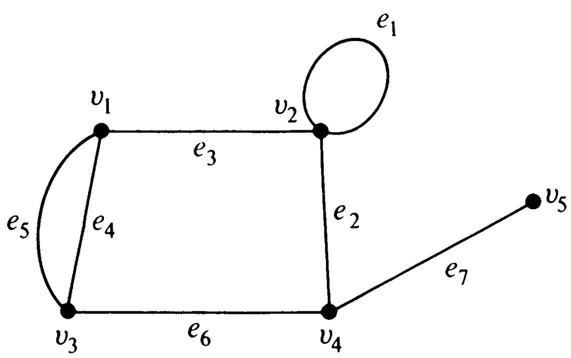

Fig. 1-1 A graph with five vertices and seven edges.

$d(v_3) = d(v_4) = 3$, $d(v_2) = 4$, and $d(v_5) = 1$. The degree of a vertex is sometimes also referred to as its *valency*.

Let us now consider a graph G with e edges and n vertices $v_1, v_2, \ldots, v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. That is,

$$\sum_{i=1}^{n} d(v_i) = 2e. \qquad (1\text{-}1)$$

Taking Fig. 1-1 as an example, once more,

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$$
$$= 3 + 4 + 3 + 3 + 1 = 14 = \text{twice the number of edges.}$$

From Eq. (1-1) we shall derive the following interesting result.

THEOREM 1-1

The number of vertices of odd degree in a graph is always even.

Proof: If we consider the vertices with odd and even degrees separately, the quantity in the left side of Eq. (1-1) can be expressed as the sum of two sums, each taken over vertices of even and odd degrees, respectively, as follows:

$$\sum_{i=1}^{n} d(v_i) = \sum_{\text{even}} d(v_j) + \sum_{\text{odd}} d(v_k). \qquad (1\text{-}2)$$

Since the left-hand side in Eq. (1-2) is even, and the first expression on the right-hand side is even (being a sum of even numbers), the second expression must also be even:

$$\sum_{\text{odd}} d(v_k) = \text{an even number.} \qquad (1\text{-}3)$$

Because in Eq. (1-3) each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum an even number. Hence the theorem. ■

A graph in which all vertices are of equal degree is called a *regular graph* (or simply a *regular*). The graph of three utilities shown in Fig. 1-7 is a regular of degree three.

1-5. ISOLATED VERTEX, PENDANT VERTEX, AND NULL GRAPH

A vertex having no incident edge is called an *isolated vertex*. In other words, isolated vertices are vertices with zero degree. Vertices v_4 and v_7 in Fig. 1-11, for example, are isolated vertices. A vertex of degree one is called

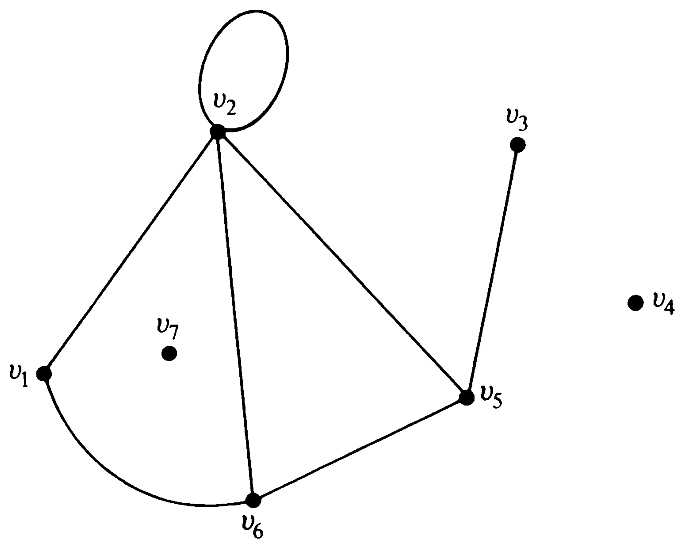

Fig. 1-11 Graph containing isolated vertices, series edges, and a pendant vertex.

a *pendant vertex* or an *end vertex*. Vertex v_3 in Fig. 1-11 is a pendant vertex. Two adjacent edges are said to be in *series* if their common vertex is of degree two. In Fig. 1-11, the two edges incident on v_1 are in series.

In the definition of a graph $G = (V, E)$, it is possible for the edge set E to be empty. Such a graph, without any edges, is called a *null graph*. In other words, every vertex in a null graph is an isolated vertex. A null graph of six vertices is shown in Fig. 1-12. Although the edge set E may be empty, the

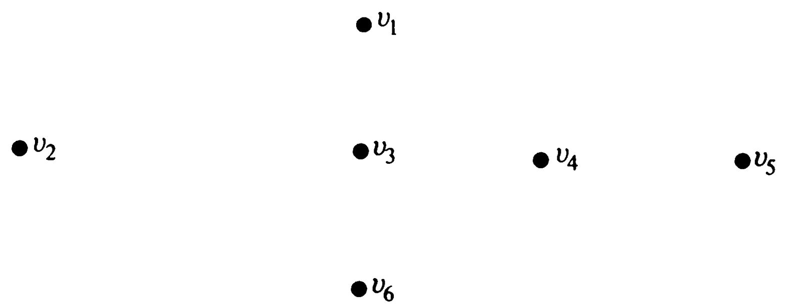

Fig. 1-12 Null graph of six vertices.

vertex set V must not be empty; otherwise, there is no graph. In other words, by definition, a graph must have at least one vertex.†

†Some authors (see, for example, [2-9], p. 1, or [15-58], p. 17) do allow the case in which the vertex set V is also empty. This they call the null graph, and they call a graph with $E = \varnothing$ and $V \neq \varnothing$ a *vertex graph*. For our purposes this distinction is of no consequence. For a lively discussion on paradoxes arising out of different definitions of the null graph, see pp. 40–41 in *Theory of Graphs: a Basis for Network Theory*, by L. M. Maxwell and M. B. Reed (Pergamon Press, N. Y. 1971).

1-6. A BRIEF HISTORY OF GRAPH THEORY

As mentioned before, graph theory was born in 1736 with Euler's paper in which he solved the Königsberg bridge problem [1-4].† For the next 100 years nothing more was done in the field.

In 1847, G. R. Kirchhoff (1824–1887) developed the theory of trees for their applications in electrical networks [1-6]. Ten years later, A. Cayley (1821–1895) discovered trees while he was trying to enumerate the isomers of saturated hydrocarbons C_nH_{2n+2} [1-3].

About the time of Kirchhoff and Cayley, two other milestones in graph theory were laid. One was the *four-color conjecture*, which states that four colors are sufficient for coloring any atlas (a map on a plane) such that the countries with common boundaries have different colors.

It is believed that A. F. Möbius (1790–1868) first presented the four-color problem in one of his lectures in 1840. About 10 years later, A. De Morgan (1806–1871) discussed this problem with his fellow mathematicians in London. De Morgan's letter is the first authenticated reference to the four-color problem. The problem became well known after Cayley published it in 1879 in the first volume of the *Proceedings of the Royal Geographic Society*. To this day, the four-color conjecture is by far the most famous unsolved problem in graph theory; it has stimulated an enormous amount of research in the field [1-11].

The other milestone is due to Sir W. R. Hamilton (1805–1865). In the year 1859 he invented a puzzle and sold it for 25 guineas to a game manufacturer in Dublin. The puzzle consisted of a wooden, regular dodecahedron (a polyhedron with 12 faces and 20 corners, each face being a regular pentagon and three edges meeting at each corner; see Fig. 2-21). The corners were marked with the names of 20 important cities: London, NewYork, Delhi, Paris, and so on. The object in the puzzle was to find a route along the edges of the dodecahedron, passing through each of the 20 cities exactly once [1-12].

Although the solution of this specific problem is easy to obtain (as we shall see in Chapter 2), to date no one has found a necessary and sufficient condition for the existence of such a route (called Hamiltonian circuit) in an arbitrary graph.

This fertile period was followed by half a century of relative inactivity. Then a resurgence of interest in graphs started during the 1920s. One of the pioneers in this period was D. König. He organized the work of other mathematicians and his own and wrote the first book on the subject, which was published in 1936 [1-7].

The past 30 years has been a period of intense activity in graph theory—both pure and applied. A great deal of research has been done and is being

†Bracketed numbers refer to references at the end of chapters.

done in this area. Thousands of papers have been published and more than a dozen books written during the past decade. Among the current leaders in the field are Claude Berge, Oystein Ore (recently deceased), Paul Erdös, William Tutte, and Frank Harary.

SUMMARY

In this chapter some basic concepts of graph theory have been introduced, and some elementary results have been obtained. An attempt has also been made to show that graphs can be used to represent almost any problem involving discrete arrangements of objects, where concern is not with the internal properties of these objects but with the relationships among them.

REFERENCES

As an elementary text on graph theory, Ore's book [1-10] is recommended. Busacker and Saaty [1-2] is a good intermediate-level book. Seshu and Reed [1-13] is specially suited for electrical engineers. Berge [1-1] and Ore [1-9] are good general texts, but are somewhat advanced. Harary's book [1-5] contains an excellent treatment of the subject. It is compact and clear, but it contains no applications and is written for an advanced student of graph theory. For relating graph theory to the rest of topology one should read [1-8], a well-written elementary book on important aspects of topology. The entertaining book of Rouse Ball [1-12] contains a variety of puzzles and games to which graphs have been applied.

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