CMS-A-DSE-B--1-TH: Operation Research (O.R.)
Lesson X: Game Theory

Introduction

Game Theory was originally developed by John Von Neumann in 1928. The mathematical relationship between game theory and linear programming was also initially recognized by Von Newmann. However, George B. Dantiz was the person to apply the simplex method successfully to solve a game theory problem. Von Neumann's book entitled "*Theory and Practice of Games and Economic Behaviour*", which he authored with Morgenstern, is considered as a pioneer work by the experts all over. It had a great impact on the development of Linear Programming and wall's Statistical Decision Theory.

Business decisions in a competitive situation do not depend on the decisions of the organization alone but on the interaction between the decisions of the organization and those of the competitors. Each firm tries to select and execute its strategies and aims to maximize its gains at the cost of its opponents. Similarly, a competitor too rises to select the best of his strategies to counteract his opponents again. Game theory deals with problems where actions and interactions of competing firms give rise to conditions of business conflict. In other words, Game Theory is a body of knowledge which is concerned with the study of decision-making in situations where two or more rational opponents are involved under conditions of competition and conflicting interest. It deals with human processes which an individual, a group, a formal or informal organization or a society, is not in complete control of the other decision-making units, the opponents, and is addressed to problems involving conflict, co-operation or both at various levels.

Some of the competitive situations in economic, social, political or military activities are:

(1) Firms trying to snatch each other's market share.

(2) Military attacks.

(3) Selection of best advertising media.

Game must be thought in a broad sense not as a kind of sport like chess or bridge but competitive situation, a kind of conflict in which one must win and the other must lose. The following are some of the fields of application of game theory:

(a) In a competitive market, sometimes companies wage a price war. What should be the bid to win major government contract in the face of competition from several contractors.

(b) An equipment dealer and a customer may be at Cross purposes regarding price but they would both want to close a mutual advantageous deal. Similarly, in a collective bargaining process, the trade union and the management of a company share the objective of striking at a mutually advantageous deal and keep the company operations going.

(c) Suppose a firm wants to introduce a new product in the market, to get a bigger share in the market, the marketing manager of the firm would be interested to know the best possible strategies of a competitor who is also trying to introduce product with different strategies e.g., price reduction, better quality, etc.

The models in the theory of games can be classified depending upon following factors:

1. Number of players

2. Sum of gains and losses

3. Strategy

Terminologies in Game Theory

The participants to the game who act as decision-makers are called players. In a game two or more participants may be in the conflict. The former of these is called a *two-person game* and the latter one is known as a *person game*. Where it does not necessarily imply that in its play exactly 'n' people would be involved, but rather that the participants can be classified into n mutually exclusive categories and members of each of the categories have identical interest.

A finite or infinite number of possible courses of action available to a player are called *strategies.* ***Example:*** Let x and y be two manufacturers and x is faced with a problem of deciding whether it is worthwhile to reduce the price of the product to counteract competition. He has two strategies:

1. reduce the price, and

2. maintain the price. Then y too has the same strategies to counter act x's strategies.

Play

A play occurs when each player selects one of his available strategies. Two basic assumptions in a play are:

(a) The choices of courses of action by players are made simultaneously.

(b) No player knows the choice of his opponents until he has decided on his own.

Outcome

Every combination of strategies of players determines an outcome called pay-off, where pay-off is nothing but a gain to a player. A loss is considered as a negative gain.

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Pay-off Matrix

The gains resulting from a game is presented in the form of a table called "pay-off matrix". A pay-off matrix comprises n rows and m columns. Where n and m indicate the number of strategies of first player and second player respectively. The pay-offs of each combination of the strategies of players are placed as elements of matrix. A positive element shows the gain to the first player (i.e., payment from II to I) and negative entry indicates the loss to the I player (i.e., payment from I to II). For instance, consider the following pay-off matrix:

	1	2	3	4	5
1	4	8	-2	6	4
2	3	6	5	3	2
3	2	-9	1	7	10

If the player chooses the first strategy and the II player uses second strategy, then the I player gains 8 units and the II player pays 8 units and similarly, if the 1 player chooses the third strategy and second player uses II strategy, then the 1 player loses 9 units and pays it to the II player and II player gains 9 units.

Strategies are classified into two types, namely, Pure strategy and Mixed Strategy.

1. A *pure Strategy* is a decision of the player to always select the same strategy.

2. A *Mixed Strategy* is a decision of the player to select more than one strategy with fixed probabilities. A mixed strategy is advantageous since the opponent is always kept guessing.

Value of the Game

The value of the game is the "expected gain to a player" if he and his opponent use their best strategies.

Saddle Point

A saddle point in a pay-off matrix corresponds to that element of the matrix which represents the 'Maxmin' value of a player and Minimax value of his opponent. For this we find Maximum element of each column and then find the Minimum value of column Maxima known as Minimax. Similarly, we identify minimum element of each row and then find the Maximum of those entries known as Maximin. If Minimax = Maximum of those entries known as Maximin.

If **Minimax = Maximin**, then **saddle point exists** and the value of the game is equal to Minimax to Maximin.

If **Minimax ≠ Maximin**, then **no saddle point exists**.

If **Minimax = Maximin**, then the **pure strategies** are called **optimum strategies.**

Usually **Maximin ≤ value of the game ≤ Minimax**.

If **Maximin = Minimax = 0**, the **game is fair**.

If **Maximin = Minimax**, the **game is strictly determinable**. The value of the game is the average pay-off that would suit the game was played over and over again.

Two Person Zero Sum Game

In a game of two persons if the algebraic sum of the gains of both the players after the game is zero, then it is called 2-person-zero-sum game.

Assumptions

1. There are 2 players having conflicting interests.

2. Each player has a finite number of strategies.

3. Each strategy selected by a player results into a certain pay-off and algebraic sum of these pay-offs to both players is zero. Two-person-zero-sum game with saddle point are called pure strategy games and two-person zero-sum game without saddle point are called mixed strategy games.

Let A and B be any 2 firms in an area have been selling a product competing for a larger share of the market. Let us assume that these firms are considering the same three strategies in a bid to gain the share in the market: Low advertising, high advertising and quality improvement and let these firms can employ only one of the strategies at a time. Under these conditions, there are 3 x 3 combinations of the moves possible and the corresponding pay-off is given below. The strategies of low advertising, high advertising and quality improvement is marked as a_1, a_2 and a_3 for the firm A and b_1, b_2 and b_3 for the firm B.

	B's Strategy				
		b_1	b_2	b_3	Row Min.
A's Strategy	a_1	12	-8	-2	-8
	a_2	6	7	3	3
	a_3	-10	-6	2	-10
Col. Max.		12	7	3	

Maximin = Maximum of Row Minimum =3

Minimax = Minimum of Column Maximum = 3

Therefore, Saddle point = (a2, b3), and

Value of the game = 3

Thus pay-off matrix is drawn from as point of view. A positive pay-off indicates that firm A has gained the market share at the expense of firm B and negative value indicates B’s gain at A’s expense. The problem now is to determine the best strategy for A and B. With the assumption that each one is not aware of the move the other is likely to take reference to the pay-off matrix if firm A employs strategy a_1, then firm B employs strategy b_2 in order to maximize its gain.

Similarly, if A’s strategies are a_2 and a_3, then B’s strategies are b_3 and b_1 respectively. Now, firm A would like to make the best use of the situation by choosing the maximum of these minimal pay-offs. Since the minimal pay-offs corresponding to a_1, a_2 and a_3 are respectively -8, 3 and 10, firm A would select a_2 as its strategy. The decision rule here is *Maximin strategy.* Similarly, if firm chooses b_1, then A will prefer a_1 and if B uses b_2 and b_3 then firm A uses the strategy a_2. To minimize the advantage occurring to A, firm B would select a strategy a_2. To minimize the advantage occurring to A, firm B would select strategy a_2. To minimize the advantage occurring to A, firm B would select a strategy that would yield the least advantage to its competitor, i.e., b_3. The decision rule here is Minimax strategy.

Here, Minimax value = Maximin value = 3, which the value of the game and corresponds to the saddle point. Saddle point can be easily obtained for 2-person pure strategy games. We shall deal with ***2-person mixed strategy games.***

Here, the players play more than one strategy and no saddle point exists. To determine the optimal strategies, the analyst needs to evaluate the probabilities (the proportion of time for which each strategy is played). For doing so we have 3 methods namely:

1. Algebraic Method

2. Iterative Method for Approximate Solution

3. Linear Programming Method.

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Algebraic Method

Let A and B be any two players with the following pay-off matrix; a_1, a_2 and b_1, b_2 denote the strategies of A and B respectively. Let P_{ij} denote the elements of pay-off matrix i, j = 1,2

Player B

Player A: Strategies	b_1	b_2
a_1	P_{11}	P_{12}
a_2	P_{21}	P_{22}

Player A has only 2 strategies namely, a_1 and a_2. If probability that he chooses a_1 is x then probability that he chooses a_2 is 1 – x. Similarly, if probability that player B chooses b_1 and b_2 is y and 1 – y respectively. Let us consider the expected gain which is the weighted average of the possible outcomes and is the product of payoff and the probabilities of the strategies. If player B plays b_1 throughout, then the gain to A is equal to

$$xP_{11} + (1-x)P_{21} \qquad \text{.................... (1)}$$

If B play b_2 throughout, then the gain A is equal to

$$xP_{12} + (1-x)P_{22} \qquad \text{.................... (2)}$$

From (1) and (2)

$$xP_{11} + (1-x)P_{21} = xP_{12} + (1-x)P_{22}$$

$$xP_{11} - xP_{12} + (1-x)P_{21} - (1-x)P_{22} = 0$$

$$x(P_{11} - P_{12}) + 1(P_{21} - P_{22}) - x(P_{21} - P_{22}) = 0$$

$$x[P_{11} - P_{12}) - (P_{21} - P_{22})] = -(P_{21} - P_{22})$$

$$x[(P_{11} - P_{21}) + (P_{22} - P_{21})] = (P_{22} - P_{21})$$

Therefore

$$x = \frac{(P_{22} - P_{21})}{(P_{11} - P_{12}) + (P_{22} - P_{21})} \qquad \text{............... (3)}$$

The value of the game i.e., gain to A from B can be obtained by substituting for x in (1) which on substitution and rearrangement becomes

$$\text{Gain} = \frac{(P_{11}P_{22} - P_{21}P_{12})}{(P_{11} - P_{12}) + (P_{22} - P_{21})}; \text{ i.e. } |P| \qquad \text{(Sum of row differences)}$$

Example: The pay-off matrix is

	B		Row Min.
A	1 5	4 3	1 3
Col. Max.	5	4	

Minimax = 4 and Maximin = 3

Here Minimax ≠ Maximin. Hence no saddle point exists. Let x and 1–x be the probability of A playing a_1, and a_2 and y and 1 – y be the probability of B playing b_1 and b_2 respectively.

We have

$$x = \frac{(P_{22} - P_{21})}{(P_{11} - P_{12}) + (P_{11} - P_{12})}$$

$$= \frac{(3-5)}{(1-4)+(3-5)}$$

$$= \frac{-2}{-5} = \frac{-2}{-3+(-2)}$$

Therefore $= \frac{2}{5}$ = Hence, $1 - x = \frac{3}{5}$

Similarly,

$$y = \frac{(P_{22} - P_{12})}{(P_{11} - P_{21}) + (P_{22} - P_{12})}$$

$$= \frac{(3-4)}{(1-5)+(3-4)}$$

$$= \frac{-1}{-4 + -1}$$

$$= \frac{-1}{-5}$$

Therefore, $y = \frac{1}{5}$

$$1 - y = \frac{4}{5}$$

Therefore Optimum mixed strategies for player B are to play first column 1/5 of the time and second column 4/5 of the time and that for A are to play first row 2/5 of the time and second row 3/5 of the time value of the game

$$V = \frac{P_{11}P_{22} - P_{12}P_{21}}{(P_{11} - P_{12}) + (P_{22} - P_{21})}$$

$$= \frac{1 \times 3 - 5 \times 4}{(P_{11} - P_{12}) + (P_{22} - P_{21})}$$

$$= \frac{3 - 20}{-3 - 2}$$

$$= \frac{-17}{-15}$$

$$= 3.4$$