Example 7.5 [4-queens] Let us see how backtracking works on the 4-queens problem of Example 7.3. As a bounding function, we use the obvious criteria that if $(x_1, x_2, \ldots, x_i)$ is the path to the current E-node, then all children nodes with parent-child labelings x_{i+1} are such that $(x_1, \ldots, x_{i+1})$ represents a chessboard configuration in which no two queens are attacking. We start with the root node as the only live node. This becomes the E-node and the path is (). We generate one child. Let us assume that the children are generated in ascending order. Thus, node number 2 of Figure 7.2 is generated and the path is now (1). This corresponds to placing queen 1 on column 1. Node 2 becomes the E-node. Node 3 is generated and immediately killed. The next node generated is node 8 and the path becomes (1, 3). Node 8 becomes the E-node. However, it gets killed as all its children represent board configurations that cannot lead to an answer node. We backtrack to node 2 and generate another child, node 13. The path is now (1, 4). Figure 7.5 shows the board configurations as backtracking proceeds. Figure 7.5 shows graphically the steps that the backtracking algorithm goes through as it tries to find a solution. The dots indicate placements of a queen which were tried and rejected because another queen was attacking. In Figure 7.5(b) the second queen is placed on columns 1 and 2 and finally settles on column 3. In Figure 7.5(c) the algorithm tries all four columns and is unable to place the next queen on a square. Backtracking now takes place. In Figure 7.5(d) the second queen is moved to the next possible column, column 4 and the third queen is placed on column 2. The boards in Figure 7.5 (e), (f), (g), and (h) show the remaining steps that the algorithm goes through until a solution is found.

Figure 7.6 shows the part of the tree of Figure 7.2 that is generated. Nodes are numbered in the order in which they are generated. A node that gets killed as a result of the bounding function has a B under it. Contrast this tree with Figure 7.2 which contains 31 nodes. □

With this example completed, we are now ready to present a precise formulation of the backtracking process. We continue to treat backtracking in a general way. We assume that all answer nodes are to be found and not just one. Let $(x_1, x_2, \ldots, x_i)$ be a path from the root to a node in a state space tree. Let $T(x_1, x_2, \ldots, x_i)$ be the set of all possible values for x_{i+1} such that $(x_1, x_2, \ldots, x_{i+1})$ is also a path to a problem state. $T(x_1, x_2, \ldots, x_n) = \emptyset$.

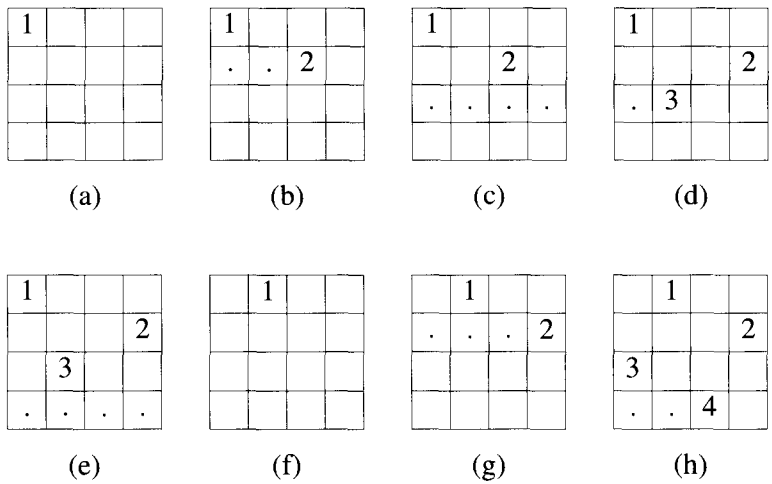

Figure 7.5 Example of a backtrack solution to the 4-queens problem

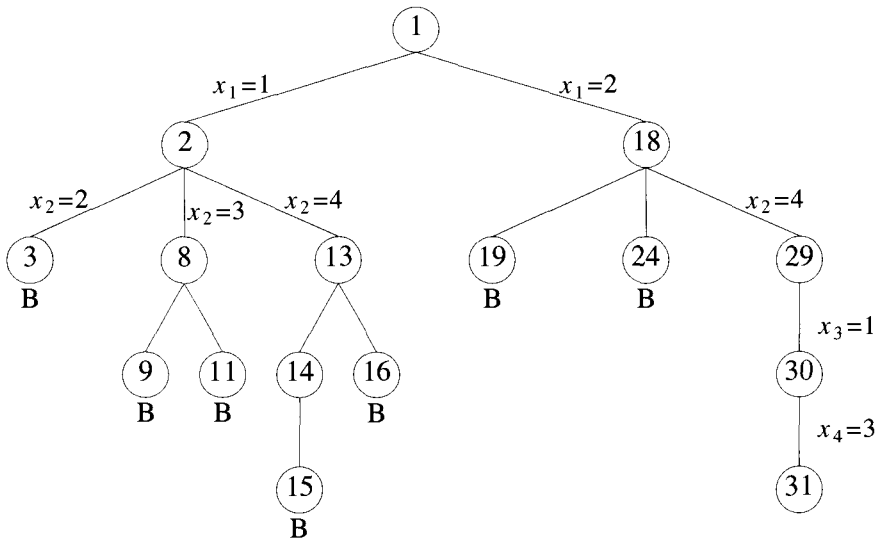

Figure 7.6 Portion of the tree of Figure 7.2 that is generated during backtracking

We assume the existence of bounding function B_{i+1} (expressed as predicates) such that if $B_{i+1}(x_1, x_2, \ldots, x_{i+1})$ is false for a path $(x_1, x_2, \ldots, x_{i+1})$ from the root node to a problem state, then the path cannot be extended to reach an answer node. Thus the candidates for position $i+1$ of the solution vector $(x_1, \ldots, x_n)$ are those values which are generated by T and satisfy B_{i+1}. Algorithm 7.1 presents a recursive formulation of the backtracking technique. It is natural to describe backtracking in this way since it is essentially a postorder traversal of a tree (see Section 6.1). This recursive version is initially invoked by

```
Backtrack(1);
```

```
1   Algorithm Backtrack(k)
2   // This schema describes the backtracking process using
3   // recursion. On entering, the first k − 1 values
4   // x[1], x[2], ..., x[k − 1] of the solution vector
5   // x[1 : n] have been assigned. x[ ] and n are global.
6   {
7       for (each x[k] ∈ T(x[1], ..., x[k − 1])) do
8       {
9           if (B_k(x[1], x[2], ..., x[k]) ≠ 0) then
10          {
11              if (x[1], x[2], ..., x[k] is a path to an answer node)
12                  then write (x[1 : k]);
13              if (k < n) then Backtrack(k + 1);
14          }
15      }
16  }
```

Algorithm 7.1 Recursive backtracking algorithm

The solution vector $(x_1, \ldots, x_n)$, is treated as a global array $x[1:n]$. All the possible elements for the kth position of the tuple that satisfy B_k are generated, one by one, and adjoined to the current vector $(x_1, \ldots, x_{k-1})$. Each time x_k is attached, a check is made to determine whether a solution has been found. Then the algorithm is recursively invoked. When the **for** loop of line 7 is exited, no more values for x_k exist and the current copy of Backtrack ends. The last unresolved call now resumes, namely, the one that continues to examine the remaining elements assuming only $k-2$ values have been set.

Note that this algorithm causes *all* solutions to be printed and assumes that tuples of various sizes may make up a solution. If only a single solution is desired, then a flag can be added as a parameter to indicate the first occurrence of success.

```
1   Algorithm IBacktrack(n)
2   // This schema describes the backtracking process.
3   // All solutions are generated in x[1 : n] and printed
4   // as soon as they are determined.
5   {
6       k := 1;
7       while (k ≠ 0) do
8       {
9           if (there remains an untried x[k] ∈ T(x[1], x[2], ...,
10              x[k − 1])  and B_k(x[1], ..., x[k]) is true) then
11          {
12                  if (x[1], ..., x[k] is a path to an answer node)
13                      then write (x[1 : k]);
14                  k := k + 1; // Consider the next set.
15          }
16          else k := k − 1; // Backtrack to the previous set.
17      }
18  }
```

Algorithm 7.2 General iterative backtracking method

An iterative version of Algorithm 7.1 appears in Algorithm 7.2. Note that $T()$ will yield the set of all possible values that can be placed as the first component x_1 of the solution vector. The component x_1 will take on those values for which the bounding function $B_1(x_1)$ is true. Also note how the elements are generated in a depth first manner. The variable k is continually incremented and a solution vector is grown until either a solution is found or no untried value of x_k remains. When k is decremented, the algorithm must resume the generation of possible elements for the kth position that have not yet been tried. Therefore one must develop a procedure that generates these values in some order. If only one solution is desired, replacing **write** $(x[1 : k])$; with **{write** $(x[1 : k])$; **return;}** suffices.

The efficiency of both the backtracking algorithms we've just seen depends very much on four factors: (1) the time to generate the next x_k, (2) the number of x_k satisfying the explicit constraints, (3) the time for the bounding functions B_k, and (4) the number of x_k satisfying the B_k. Bound-

```
1   Algorithm Place(k, i)
2   // Returns true if a queen can be placed in kth row and
3   // ith column. Otherwise it returns false. x[ ] is a
4   // global array whose first (k − 1) values have been set.
5   // Abs(r) returns the absolute value of r.
6   {
7       for j := 1 to k − 1 do
8           if ((x[j] = i) // Two in the same column
9                or (Abs(x[j] − i) = Abs(j − k)))
10                   // or in the same diagonal
11               then return false;
12      return true;
13  }
```

Algorithm 7.4 Can a new queen be placed?

```
1   Algorithm NQueens(k, n)
2   // Using backtracking, this procedure prints all
3   // possible placements of n queens on an n × n
4   // chessboard so that they are nonattacking.
5   {
6       for i := 1 to n do
7       {
8           if Place(k, i) then
9           {
10              x[k] := i;
11              if (k = n) then write (x[1 : n]);
12              else NQueens(k + 1, n);
13          }
14      }
15  }
```

Algorithm 7.5 All solutions to the n-queens problem

At this point we might wonder how effective function NQueens is over the brute force approach. For an 8×8 chessboard there are $\binom{64}{8}$ possible ways to place 8 pieces, or approximately 4.4 billion 8-tuples to examine. However, by allowing only placements of queens on distinct rows and columns, we require the examination of at most 8!, or only 40,320 8-tuples.

We can use Estimate to estimate the number of nodes that will be generated by NQueens. Note that the assumptions that are needed for Estimate do hold for NQueens. The bounding function is static. No change is made to the function as the search proceeds. In addition, all nodes on the same level of the state space tree have the same degree. In Figure 7.8 we see five 8×8 chessboards that were created using Estimate.

As required, the placement of each queen on the chessboard was chosen randomly. With each choice we kept track of the number of columns a queen could legitimately be placed on. These numbers are listed in the vector beneath each chessboard. The number following the vector represents the value that function Estimate would produce from these sizes. The average of these five trials is 1625. The total number of nodes in the 8-queens state space tree is

$$1 + \sum_{j=0}^{7} \left[\Pi_{i=0}^{j} (8 - i) \right] = 69,281$$

So the estimated number of unbounded nodes is only about 2.34% of the total number of nodes in the 8-queens state space tree. (See the exercises for more ideas about the efficiency of NQueens.)

Reference :

Text Book-

E. Horowitz and S. Sahni: Fundamental of Computer Algorithms, Galgotia Pub. /Pitman, New Delhi/London, 1987/1978.