7.6 KNAPSACK PROBLEM

In this section we reconsider a problem that was defined and solved by a dynamic programming algorithm in Chapter 5, the 0/1 knapsack optimization problem. Given n positive weights w_i, n positive profits p_i, and a positive number m that is the knapsack capacity, this problem calls for choosing a subset of the weights such that

$$\sum_{1 \le i \le n} w_i x_i \le m \quad \text{and} \quad \sum_{1 \le i \le n} p_i x_i \text{ is maximized} \tag{7.2}$$

The x_i's constitute a zero-one-valued vector.

The solution space for this problem consists of the 2^n distinct ways to assign zero or one values to the x_i's. Thus the solution space is the same as that for the sum of subsets problem. Two possible tree organizations are possible. One corresponds to the fixed tuple size formulation (Figure 7.4) and the other to the variable tuple size formulation (Figure 7.3). Backtracking algorithms for the knapsack problem can be arrived at using either of these two state space trees. Regardless of which is used, bounding functions are needed to help kill some live nodes without expanding them. A good bounding function for this problem is obtained by using an upper bound on the value of the best feasible solution obtainable by expanding the given live node and any of its descendants. If this upper bound is not higher than

the value of the best solution determined so far, then that live node can be killed.

We continue the discussion using the fixed tuple size formulation. If at node Z the values of x_i, $1 \leq i \leq k$, have already been determined, then an upper bound for Z can be obtained by relaxing the requirement $x_i = 0$ or 1 to $0 \leq x_i \leq 1$ for $k+1 \leq i \leq n$ and using the greedy algorithm of Section 4.2 to solve the relaxed problem. Function Bound(cp, cw, k) (Algorithm 7.11) determines an upper bound on the best solution obtainable by expanding any node Z at level $k+1$ of the state space tree. The object weights and profits are $w[i]$ and $p[i]$. It is assumed that $p[i]/w[i] \geq p[i+1]/w[i+1]$, $1 \leq i < n$.

```
1   Algorithm Bound(cp, cw, k)
2   // cp is the current profit total, cw is the current
3   // weight total; k is the index of the last removed
4   // item; and m is the knapsack size.
5   {
6       b := cp; c := cw;
7       for i := k + 1 to n do
8       {
9           c := c + w[i];
9           if (c < m) then b := b + p[i];
10          else return b + (1 − (c − m)/w[i]) ∗ p[i];
11      }
12      return b;
13  }
```

Algorithm 7.11 A bounding function

From Bound it follows that the bound for a feasible left child of a node Z is the same as that for Z. Hence, the bounding function need not be used whenever the backtracking algorithm makes a move to the left child of a node. The resulting algorithm is BKnap (Algorithm 7.12). It was obtained from the recursive backtracking schema. Initially set $fp := -1$;. This algorithm is invoked as

```
BKnap(1, 0, 0);
```

When $fp \neq -1$, $x[i]$, $1 \leq i \leq n$, is such that $\sum_{i=1}^{n} p[i]x[i] = fp$. In lines 8 to 18 left children are generated. In line 20, Bound is used to test whether a

```
1   Algorithm BKnap(k, cp, cw)
2   // m is the size of the knapsack; n is the number of weights
3   // and profits. w[ ] and p[ ] are the weights and profits.
4   // p[i]/w[i] ≥ p[i + 1]/w[i + 1]. fw is the final weight of
5   // knapsack; fp is the final maximum profit. x[k] = 0 if w[k]
6   // is not in the knapsack; else x[k] = 1.
7   {
8       // Generate left child.
9       if (cw + w[k] ≤ m) then
10      {
11          y[k] := 1;
12          if (k < n) then BKnap(k + 1, cp + p[k], cw + w[k]);
13          if ((cp + p[k] > fp) and (k = n)) then
14          {
15              fp := cp + p[k]; fw := cw + w[k];
16              for j := 1 to k do x[j] := y[j];
17          }
18      }
19      // Generate right child.
20      if (Bound(cp, cw, k) ≥ fp) then
21      {
22          y[k] := 0; if (k < n) then BKnap(k + 1, cp, cw);
23          if ((cp > fp) and (k = n)) then
24          {
25              fp := cp; fw := cw;
26              for j := 1 to k do x[j] := y[j];
27          }
28      }
29  }
```

Algorithm 7.12 Backtracking solution to the 0/1 knapsack problem

right child should be generated. The path $y[i]$, $1 \leq i \leq k$, is the path to the current node. The current weight $cw = \sum_{i=1}^{k-1} w[i]y[i]$ and $cp = \sum_{i=1}^{k-1} p[i]y[i]$. In lines 13 to 17 and 23 to 27 the solution vector is updated if need be.

So far, all our backtracking algorithms have worked on a static state space tree. We now see how a dynamic state space tree can be used for the knapsack problem. One method for dynamically partitioning the solution space is based on trying to obtain an optimal solution using the greedy algorithm of Section 4.2. We first replace the integer constraint $x_i = 0$ or 1 by the constraint $0 \leq x_i \leq 1$. This yields the relaxed problem

$$\max \sum_{1 \leq i \leq n} p_i x_i \quad \text{subject to} \quad \sum_{1 \leq i \leq n} w_i x_i \leq m \tag{7.3}$$

$$0 \leq x_i \leq 1, \qquad 1 \leq i \leq n$$

If the solution generated by the greedy method has all x_i's equal to zero or one, then it is also an optimal solution to the original 0/1 knapsack problem. If this is not the case, then exactly one x_i will be such that $0 < x_i < 1$. We partition the solution space of (7.2) into two subspaces. In one $x_i = 0$ and in the other $x_i = 1$. Thus the left subtree of the state space tree will correspond to $x_i = 0$ and the right to $x_i = 1$. In general, at each node Z of the state space tree the greedy algorithm is used to solve (7.3) under the added restrictions corresponding to the assignments already made along the path from the root to this node. In case the solution is all integer, then an optimal solution for this node has been found. If not, then there is exactly one x_i such that $0 < x_i < 1$. The left child of Z corresponds to $x_i = 0$, and the right to $x_i = 1$.

The justification for this partitioning scheme is that the noninteger x_i is what prevents the greedy solution from being a feasible solution to the 0/1 knapsack problem. So, we would expect to reach a feasible greedy solution quickly by forcing this x_i to be integer. Choosing left branches to correspond to $x_i = 0$ rather than $x_i = 1$ is also justifiable. Since the greedy algorithm requires $p_j/w_j \geq p_{j+1}/w_{j+1}$, we would expect most objects with low index (i.e., small j and hence high density) to be in an optimal filling of the knapsack. When x_i is set to zero, we are not preventing the greedy algorithm from using any of the objects with $j < i$ (unless x_j has already been set to zero). On the other hand, when x_i is set to one, some of the x_j's with $j < i$ will not be able to get into the knapsack. Therefore we expect to arrive at an optimal solution with $x_i = 0$. So we wish the backtracking algorithm to try this alternative first. Hence the left subtree corresponds to $x_i = 0$.

Example 7.7 Let us try out a backtracking algorithm and the above dynamic partitioning scheme on the following data: $p = \{11, 21, 31, 33, 43, 53, 55, 65\}$, $w = \{1, 11, 21, 23, 33, 43, 45, 55\}$, $m = 110$, and $n = 8$. The greedy

solution corresponding to the root node (i.e., Equation (7.3)) is $x = \{1, 1, 1, 1, 1, 21/45, 0, 0\}$. Its value is 164.88. The two subtrees of the root correspond to $x_6 = 0$ and $x_6 = 1$, respectively (Figure 7.16). The greedy solution at node 2 is $x = \{1, 1, 1, 1, 1, 0, 21/45, 0\}$. Its value is 164.66. The solution space at node 2 is partitioned using $x_7 = 0$ and $x_7 = 1$. The next E-node is node 3. The solution here has $x_8 = 21/55$. The partitioning now is with $x_8 = 0$ and $x_8 = 1$. The solution at node 4 is all integer so there is no need to expand this node further. The best solution found so far has value 139 and $x = \{1, 1, 1, 1, 1, 0, 0, 0\}$. Node 5 is the next E-node. The greedy solution for this node is $x = \{1, 1, 1, 22/23, 0, 0, 0, 1\}$. Its value is 159.56. The partitioning is now with $x_4 = 0$ and $x_4 = 1$. The greedy solution at node 6 has value 156.66 and $x_5 = 2/3$. Next, node 7 becomes the E-node. The solution here is $\{1, 1, 1, 0, 0, 0, 0, 1\}$. Its value is 128. Node 7 is not expanded as the greedy solution here is all integer. At node 8 the greedy solution has value 157.71 and $x_3 = 4/7$. The solution at node 9 is all integer and has value 140. The greedy solution at node 10 is $\{1, 0, 1, 0, 1, 0, 0, 1\}$. Its value is 150. The next E-node is 11. Its value is 159.52 and $x_3 = 20/21$. The partitioning is now on $x_3 = 0$ and $x_3 = 1$. The remainder of the backtracking process on this knapsack instance is left as an exercise. □

Experimental work due to E. Horowitz and S. Sahni, cited in the references, indicates that backtracking algorithms for the knapsack problem generally work in less time when using a static tree than when using a dynamic tree. The dynamic partitioning scheme is, however, useful in the solution of integer linear programs. The general integer linear program is mathematically stated in (7.4).

$$\begin{aligned} &\text{minimize} \quad \textstyle\sum_{1 \le j \le n} c_j \, x_j \\ &\text{subject to} \quad \textstyle\sum_{1 \le j \le n} a_{ij} \, x_j \le b_i, \quad 1 \le i \le m \\ &\qquad x_j's \text{ are nonnegative integers} \end{aligned} \tag{7.4}$$

If the integer constraints on the x_i's in (7.4) are replaced by the constraint $x_i \ge 0$, then we obtain a linear program whose optimal solution has a value at least as large as the value of an optimal solution to (7.4). Linear programs can be solved using the simplex methods (see the references). If the solution is not all integer, then a noninteger x_i is chosen to partition the solution space. Let us assume that the value of x_i in the optimal solution to the linear program corresponding to any node Z in the state space is v and v is not an integer. The left child of Z corresponds to $x_i \le \lfloor v \rfloor$ whereas the right child of Z correspond to $x_i \ge \lceil v \rceil$. Since the resulting state space tree has a potentially infinite depth (note that on the path from the root to a node Z

the solution space can be partitioned on one x_i many times as each x_i can have as value any nonnegative integer), it is almost always searched using a branch-and-bound method (see Chapter 8).

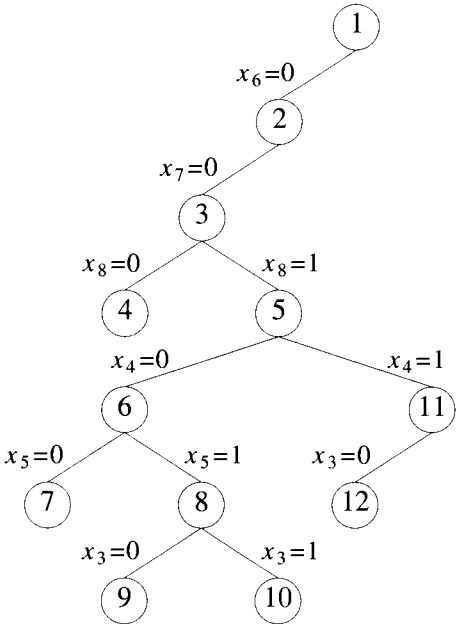

Figure 7.16 Part of the dynamic state space tree generated in Example 7.7

Reference :

Text Book-

E. Horowitz and S. Sahni: Fundamental of Computer Algorithms, Galgotia Pub. /Pitman, New Delhi/London, 1987/1978.