
15.5 Optimal binary search trees

Suppose that we are designing a program to translate text from English to French. For each occurrence of each English word in the text, we need to look up its French equivalent. We could perform these lookup operations by building a binary search tree with n English words as keys and their French equivalents as satellite data. Because we will search the tree for each individual word in the text, we want the total time spent searching to be as low as possible. We could ensure an $O(\lg n)$ search time per occurrence by using a red-black tree or any other balanced binary search tree. Words appear with different frequencies, however, and a frequently used word such as *the* may appear far from the root while a rarely used word such as *machicolation* appears near the root. Such an organization would slow down the translation, since the number of nodes visited when searching for a key in a binary search tree equals one plus the depth of the node containing the key. We want words that occur frequently in the text to be placed nearer the root.⁶ Moreover, some words in the text might have no French translation,⁷ and such words would not appear in the binary search tree at all. How do we organize a binary search tree so as to minimize the number of nodes visited in all searches, given that we know how often each word occurs?

What we need is known as an *optimal binary search tree*. Formally, we are given a sequence $K = \langle k_1, k_2, \dots, k_n \rangle$ of n distinct keys in sorted order (so that $k_1 < k_2 < \dots < k_n$), and we wish to build a binary search tree from these keys. For each key k_i , we have a probability p_i that a search will be for k_i . Some searches may be for values not in K , and so we also have $n + 1$ “dummy keys”

⁶If the subject of the text is castle architecture, we might want *machicolation* to appear near the root.

⁷Yes, *machicolation* has a French counterpart: *mâchicoulis*.

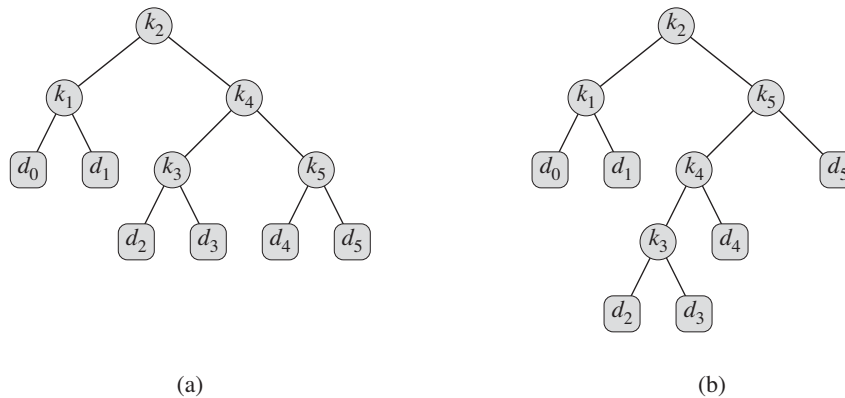


Figure 15.9 Two binary search trees for a set of $n = 5$ keys with the following probabilities:

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

(a) A binary search tree with expected search cost 2.80. (b) A binary search tree with expected search cost 2.75. This tree is optimal.

$d_0, d_1, d_2, \dots, d_n$ representing values not in K . In particular, d_0 represents all values less than k_1 , d_n represents all values greater than k_n , and for $i = 1, 2, \dots, n-1$, the dummy key d_i represents all values between k_i and k_{i+1} . For each dummy key d_i , we have a probability q_i that a search will correspond to d_i . Figure 15.9 shows two binary search trees for a set of $n = 5$ keys. Each key k_i is an internal node, and each dummy key d_i is a leaf. Every search is either successful (finding some key k_i) or unsuccessful (finding some dummy key d_i), and so we have

$$\sum_{i=1}^n p_i + \sum_{i=0}^n q_i = 1. \quad (15.10)$$

Because we have probabilities of searches for each key and each dummy key, we can determine the expected cost of a search in a given binary search tree T . Let us assume that the actual cost of a search equals the number of nodes examined, i.e., the depth of the node found by the search in T , plus 1. Then the expected cost of a search in T is

$$\begin{aligned} E[\text{search cost in } T] &= \sum_{i=1}^n (\text{depth}_T(k_i) + 1) \cdot p_i + \sum_{i=0}^n (\text{depth}_T(d_i) + 1) \cdot q_i \\ &= 1 + \sum_{i=1}^n \text{depth}_T(k_i) \cdot p_i + \sum_{i=0}^n \text{depth}_T(d_i) \cdot q_i, \end{aligned} \quad (15.11)$$

where depth_T denotes a node's depth in the tree T . The last equality follows from equation (15.10). In Figure 15.9(a), we can calculate the expected search cost node by node:

node	depth	probability	contribution
k_1	1	0.15	0.30
k_2	0	0.10	0.10
k_3	2	0.05	0.15
k_4	1	0.10	0.20
k_5	2	0.20	0.60
d_0	2	0.05	0.15
d_1	2	0.10	0.30
d_2	3	0.05	0.20
d_3	3	0.05	0.20
d_4	3	0.05	0.20
d_5	3	0.10	0.40
Total			2.80

For a given set of probabilities, we wish to construct a binary search tree whose expected search cost is smallest. We call such a tree an *optimal binary search tree*. Figure 15.9(b) shows an optimal binary search tree for the probabilities given in the figure caption; its expected cost is 2.75. This example shows that an optimal binary search tree is not necessarily a tree whose overall height is smallest. Nor can we necessarily construct an optimal binary search tree by always putting the key with the greatest probability at the root. Here, key k_5 has the greatest search probability of any key, yet the root of the optimal binary search tree shown is k_2 . (The lowest expected cost of any binary search tree with k_5 at the root is 2.85.)

As with matrix-chain multiplication, exhaustive checking of all possibilities fails to yield an efficient algorithm. We can label the nodes of any n -node binary tree with the keys k_1, k_2, \dots, k_n to construct a binary search tree, and then add in the dummy keys as leaves. In Problem 12-4, we saw that the number of binary trees with n nodes is $\Omega(4^n/n^{3/2})$, and so we would have to examine an exponential number of binary search trees in an exhaustive search. Not surprisingly, we shall solve this problem with dynamic programming.

Step 1: The structure of an optimal binary search tree

To characterize the optimal substructure of optimal binary search trees, we start with an observation about subtrees. Consider any subtree of a binary search tree. It must contain keys in a contiguous range k_i, \dots, k_j , for some $1 \leq i \leq j \leq n$. In addition, a subtree that contains keys k_i, \dots, k_j must also have as its leaves the dummy keys d_{i-1}, \dots, d_j .

Now we can state the optimal substructure: if an optimal binary search tree T has a subtree T' containing keys k_i, \dots, k_j , then this subtree T' must be optimal as

well for the subproblem with keys k_i, \dots, k_j and dummy keys d_{i-1}, \dots, d_j . The usual cut-and-paste argument applies. If there were a subtree T'' whose expected cost is lower than that of T' , then we could cut T' out of T and paste in T'' , resulting in a binary search tree of lower expected cost than T , thus contradicting the optimality of T .

We need to use the optimal substructure to show that we can construct an optimal solution to the problem from optimal solutions to subproblems. Given keys k_i, \dots, k_j , one of these keys, say k_r ($i \leq r \leq j$), is the root of an optimal subtree containing these keys. The left subtree of the root k_r contains the keys k_i, \dots, k_{r-1} (and dummy keys d_{i-1}, \dots, d_{r-1}), and the right subtree contains the keys k_{r+1}, \dots, k_j (and dummy keys d_r, \dots, d_j). As long as we examine all candidate roots k_r , where $i \leq r \leq j$, and we determine all optimal binary search trees containing k_i, \dots, k_{r-1} and those containing k_{r+1}, \dots, k_j , we are guaranteed that we will find an optimal binary search tree.

There is one detail worth noting about “empty” subtrees. Suppose that in a subtree with keys k_i, \dots, k_j , we select k_i as the root. By the above argument, k_i 's left subtree contains the keys k_i, \dots, k_{i-1} . We interpret this sequence as containing no keys. Bear in mind, however, that subtrees also contain dummy keys. We adopt the convention that a subtree containing keys k_i, \dots, k_{i-1} has no actual keys but does contain the single dummy key d_{i-1} . Symmetrically, if we select k_j as the root, then k_j 's right subtree contains the keys k_{j+1}, \dots, k_j ; this right subtree contains no actual keys, but it does contain the dummy key d_j .

Step 2: A recursive solution

We are ready to define the value of an optimal solution recursively. We pick our subproblem domain as finding an optimal binary search tree containing the keys k_i, \dots, k_j , where $i \geq 1$, $j \leq n$, and $j \geq i - 1$. (When $j = i - 1$, there are no actual keys; we have just the dummy key d_{i-1} .) Let us define $e[i, j]$ as the expected cost of searching an optimal binary search tree containing the keys k_i, \dots, k_j . Ultimately, we wish to compute $e[1, n]$.

The easy case occurs when $j = i - 1$. Then we have just the dummy key d_{i-1} . The expected search cost is $e[i, i - 1] = q_{i-1}$.

When $j \geq i$, we need to select a root k_r from among k_i, \dots, k_j and then make an optimal binary search tree with keys k_i, \dots, k_{r-1} as its left subtree and an optimal binary search tree with keys k_{r+1}, \dots, k_j as its right subtree. What happens to the expected search cost of a subtree when it becomes a subtree of a node? The depth of each node in the subtree increases by 1. By equation (15.11), the expected search cost of this subtree increases by the sum of all the probabilities in the subtree. For a subtree with keys k_i, \dots, k_j , let us denote this sum of probabilities as

$$w(i, j) = \sum_{l=i}^j p_l + \sum_{l=i-1}^j q_l . \quad (15.12)$$

Thus, if k_r is the root of an optimal subtree containing keys k_i, \dots, k_j , we have

$$e[i, j] = p_r + (e[i, r-1] + w(i, r-1)) + (e[r+1, j] + w(r+1, j)) .$$

Noting that

$$w(i, j) = w(i, r-1) + p_r + w(r+1, j) ,$$

we rewrite $e[i, j]$ as

$$e[i, j] = e[i, r-1] + e[r+1, j] + w(i, j) . \quad (15.13)$$

The recursive equation (15.13) assumes that we know which node k_r to use as the root. We choose the root that gives the lowest expected search cost, giving us our final recursive formulation:

$$e[i, j] = \begin{cases} q_{i-1} & \text{if } j = i - 1 , \\ \min_{i \leq r \leq j} \{e[i, r-1] + e[r+1, j] + w(i, j)\} & \text{if } i \leq j . \end{cases} \quad (15.14)$$

The $e[i, j]$ values give the expected search costs in optimal binary search trees. To help us keep track of the structure of optimal binary search trees, we define $root[i, j]$, for $1 \leq i \leq j \leq n$, to be the index r for which k_r is the root of an optimal binary search tree containing keys k_i, \dots, k_j . Although we will see how to compute the values of $root[i, j]$, we leave the construction of an optimal binary search tree from these values as Exercise 15.5-1.

Step 3: Computing the expected search cost of an optimal binary search tree

At this point, you may have noticed some similarities between our characterizations of optimal binary search trees and matrix-chain multiplication. For both problem domains, our subproblems consist of contiguous index subranges. A direct, recursive implementation of equation (15.14) would be as inefficient as a direct, recursive matrix-chain multiplication algorithm. Instead, we store the $e[i, j]$ values in a table $e[1..n+1, 0..n]$. The first index needs to run to $n+1$ rather than n because in order to have a subtree containing only the dummy key d_n , we need to compute and store $e[n+1, n]$. The second index needs to start from 0 because in order to have a subtree containing only the dummy key d_0 , we need to compute and store $e[1, 0]$. We use only the entries $e[i, j]$ for which $j \geq i - 1$. We also use a table $root[i, j]$, for recording the root of the subtree containing keys k_i, \dots, k_j . This table uses only the entries for which $1 \leq i \leq j \leq n$.

We will need one other table for efficiency. Rather than compute the value of $w(i, j)$ from scratch every time we are computing $e[i, j]$ —which would take

$\Theta(j - i)$ additions—we store these values in a table $w[1..n + 1, 0..n]$. For the base case, we compute $w[i, i - 1] = q_{i-1}$ for $1 \leq i \leq n + 1$. For $j \geq i$, we compute

$$w[i, j] = w[i, j - 1] + p_j + q_j. \quad (15.15)$$

Thus, we can compute the $\Theta(n^2)$ values of $w[i, j]$ in $\Theta(1)$ time each.

The pseudocode that follows takes as inputs the probabilities p_1, \dots, p_n and q_0, \dots, q_n and the size n , and it returns the tables e and $root$.

OPTIMAL-BST(p, q, n)

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1  let  $e[1..n + 1, 0..n]$ ,  $w[1..n + 1, 0..n]$ ,
   and  $root[1..n, 1..n]$  be new tables
2  for  $i = 1$  to  $n + 1$ 
3       $e[i, i - 1] = q_{i-1}$ 
4       $w[i, i - 1] = q_{i-1}$ 
5  for  $l = 1$  to  $n$ 
6      for  $i = 1$  to  $n - l + 1$ 
7           $j = i + l - 1$ 
8           $e[i, j] = \infty$ 
9           $w[i, j] = w[i, j - 1] + p_j + q_j$ 
10         for  $r = i$  to  $j$ 
11              $t = e[i, r - 1] + e[r + 1, j] + w[i, j]$ 
12             if  $t < e[i, j]$ 
13                  $e[i, j] = t$ 
14                  $root[i, j] = r$ 
15  return  $e$  and  $root$ 
```

From the description above and the similarity to the MATRIX-CHAIN-ORDER procedure in Section 15.2, you should find the operation of this procedure to be fairly straightforward. The **for** loop of lines 2–4 initializes the values of $e[i, i - 1]$ and $w[i, i - 1]$. The **for** loop of lines 5–14 then uses the recurrences (15.14) and (15.15) to compute $e[i, j]$ and $w[i, j]$ for all $1 \leq i \leq j \leq n$. In the first iteration, when $l = 1$, the loop computes $e[i, i]$ and $w[i, i]$ for $i = 1, 2, \dots, n$. The second iteration, with $l = 2$, computes $e[i, i + 1]$ and $w[i, i + 1]$ for $i = 1, 2, \dots, n - 1$, and so forth. The innermost **for** loop, in lines 10–14, tries each candidate index r to determine which key k_r to use as the root of an optimal binary search tree containing keys k_i, \dots, k_j . This **for** loop saves the current value of the index r in $root[i, j]$ whenever it finds a better key to use as the root.

Figure 15.10 shows the tables $e[i, j]$, $w[i, j]$, and $root[i, j]$ computed by the procedure OPTIMAL-BST on the key distribution shown in Figure 15.9. As in the matrix-chain multiplication example of Figure 15.5, the tables are rotated to make

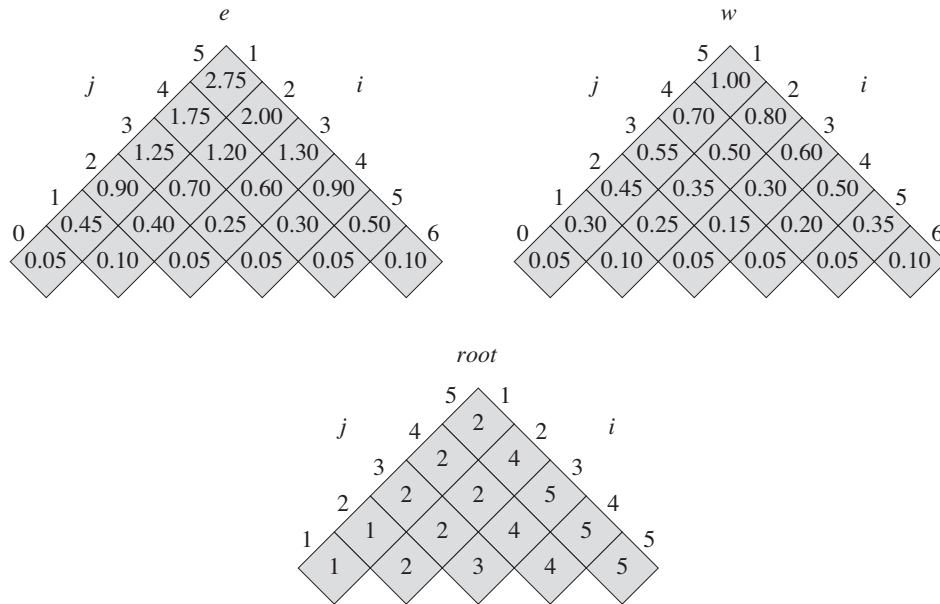


Figure 15.10 The tables $e[i, j]$, $w[i, j]$, and $root[i, j]$ computed by OPTIMAL-BST on the key distribution shown in Figure 15.9. The tables are rotated so that the diagonals run horizontally.

the diagonals run horizontally. OPTIMAL-BST computes the rows from bottom to top and from left to right within each row.

The OPTIMAL-BST procedure takes $\Theta(n^3)$ time, just like MATRIX-CHAIN-ORDER. We can easily see that its running time is $O(n^3)$, since its **for** loops are nested three deep and each loop index takes on at most n values. The loop indices in OPTIMAL-BST do not have exactly the same bounds as those in MATRIX-CHAIN-ORDER, but they are within at most 1 in all directions. Thus, like MATRIX-CHAIN-ORDER, the OPTIMAL-BST procedure takes $\Omega(n^3)$ time.

Reference :

Text Book-T. H. Cormen, C. E. Leiserson, R. L. Rivest and C. Stein : Introduction to Algorithms, Third Edition, The MIT Press Cambridge, Massachusetts London, England.