Reduction of second order linear equations to canonical forms

Let the general second order linear PDE in two variables be

\[ L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \]  

or,

\[ au_{xx} + 2bu_{xy} + cu_{yy} = \Phi(x,y,u,u_x, u_y) \] \hspace{1cm} (1a)

and let \((\xi, \eta) = (\xi(x,y), \eta(x,y))\) be a nonsingular transformation.

We write

\[ w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)). \] \hspace{1cm} (2)

where \(w\) is a solution of a second-order equation of the same type. Using the chain rule one finds that

\[ u_x = w_\xi \xi_x + w_\eta \eta_x , \]
\[ u_y = w_\xi \xi_y + w_\eta \eta_y , \]
\[ u_{xx} = w_{\xi\xi} \xi_x^2 + 2w_\xi \eta_x \xi_x \eta_x + w_{\eta\eta} \eta^2_x + w_\xi \xi_{xx} + w_\eta \eta_{xx} , \]
\[ u_{xy} = w_{\xi\xi} \xi_x \xi_y + w_\xi (\xi_x \eta_y + \xi_y \eta_x) + w_{\eta\eta} \eta_x \eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy} , \]
\[ u_{yy} = w_{\xi\xi} \xi_y^2 + 2w_\xi \eta_y \xi_y \eta_y + w_{\eta\eta} \eta^2_y + w_\xi \xi_{yy} + w_\eta \eta_{yy} . \] \hspace{1cm} (3)

Substituting these formulas into (1), we see that \(w\) satisfies the following linear PDE after transformation:

\[ \ell[w] = Aw_{\xi\xi} + 2Bw_\xi \eta_x + Cw_{\eta\eta} + Dw_\xi + Ew_\eta + Fw = G. \] \hspace{1cm} (4)

or,

\[ \ell_0[w] = Aw_{\xi\xi} + 2Bw_\xi \eta_x + Cw_{\eta\eta} = \Phi(\xi, \eta, w, w_\xi, w_\eta) \] \hspace{1cm} (4a)

where \(\Phi\) becomes \(\varphi\) and the new coefficients of the principal part of the linear operator \(\ell\) are given by

\[ A(\xi, \eta) = a \xi_x^2 + 2b \xi_x \eta_x + c \eta_x^2 , \]
\[ B(\xi, \eta) = a \xi_x \eta_x + b (\xi_x \eta_y + \xi_y \eta_x) + c \xi_y \eta_y , \]
\[ C(\xi, \eta) = a \eta_x^2 + 2b \eta_x \eta_y + c \eta_y^2 . \] \hspace{1cm} (5)

Canonical form of hyperbolic equations

**Theorem 1.** Suppose that (1) is hyperbolic in a domain \(D\). There exists a coordinate system \((\xi, \eta)\) in which the equation has the canonical form

\[ w_{\xi\eta} + \ell_1[w] = G(\xi, \eta) . \] \hspace{1cm} (6)

or,

\[ w_{\xi\eta} = \psi(\xi, \eta, w, w_\xi, w_\eta) \] \hspace{1cm} (6a)
where \( w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)) \), \( \ell_1 \) or \( \psi \) is a first-order linear differential operator, and \( G \) is a function which depends on (1).

**Proof.** Without loss of generality, we may assume that \( a(x, y) \neq 0 \) for all \( x, y \in D \). We need to find two functions \( \xi = \xi(x, y), \eta = \eta(x, y) \) such that

\[
A(\xi, \eta) = a\xi_x^2 + 2b\xi_x \xi_y + c\eta_y^2 = 0, \tag{7a}
\]

\[
C(\xi, \eta) = a\eta_x^2 + 2b\eta_x \eta_y + c\xi_y^2 = 0. \tag{7b}
\]

Dividing equation (7a) and (7b) throughout by \( \xi_y^2 \) and \( \eta_x^2 \) respectively to obtain

\[
a\left(\frac{\xi_x}{\xi_y}\right)^2 + 2b\left(\frac{\xi_x}{\xi_y}\right) + c = 0 \tag{8a}
\]

\[
a\left(\frac{\eta_x}{\eta_y}\right)^2 + 2b\left(\frac{\eta_x}{\eta_y}\right) + c = 0 \tag{8b}
\]

Equation (8a) is a quadratic equation for \( \frac{\xi_x}{\xi_y} \) whose roots are given by

\[
\mu_1(x, y) = \frac{-b - \sqrt{b^2 - ac}}{a}
\]

\[
\mu_2(x, y) = \frac{-b + \sqrt{b^2 - ac}}{a}
\]

The equation that was obtained for the function \( \eta \) (equation (7b) or (8b)) is actually the same equation as for \( \xi \), and can be found in an identical manner; therefore, we need to solve only one equation (7a) or (8a), and only two distinct roots are possible between the two equations (8a) and (8b) or (7a) and (7b).

Equation (7a) is a first-order PDE and as a quadratic form in \( \xi \) it is possible to write it as a product of two linear terms

\[
\frac{1}{a} [a\xi_x + (b + \sqrt{b^2 - ac})\xi_y][a\xi_x + (b - \sqrt{b^2 - ac})\xi_y] = 0. \tag{9a}
\]

Similarly (7b) is a quadratic form in \( \eta \) and may be written as a product of two similar linear terms

\[
\frac{1}{a} [a\eta_x + (b + \sqrt{b^2 - ac})\eta_y][a\eta_x + (b - \sqrt{b^2 - ac})\eta_y] = 0 \tag{9b}
\]

Also we may consider \( \mu_1 \) as the root of (8a) and \( \mu_2 \) as that of (8b). That is,

\[
\mu_1(x, y) = \frac{\xi_x}{\xi_y} = \frac{-b - \sqrt{b^2 - ac}}{a} \tag{10a}
\]

\[
\mu_2(x, y) = \frac{\eta_x}{\eta_y} = \frac{-b + \sqrt{b^2 - ac}}{a} \tag{10b}
\]

Therefore, we need to solve the following two first order linear equations (obtained from (9a,b)):

\[
a\xi_x + (b + \sqrt{b^2 - ac})\xi_y = 0 \tag{11a}
\]
\[ a\eta_x + (b - \sqrt{b^2 - ac})\eta_y = 0 \]  
(11b)

or equivalently (obtained from (10a,b))

\[ \xi_x - \mu_1(x, y)\xi_y = 0 \]  
(12a)

\[ \eta_x - \mu_2(x, y)\eta_y = 0 \]  
(12b)

These are the equations that define the new coordinate variables \( \xi \) and \( \eta \) that are necessary to make \( A = C = 0 \) in (4). In order to obtain a nonsingular transformation \( (\xi(x, y), \eta(x, y)) \) we choose \( \xi \) to be a solution of (11a) or (12a) and \( \eta \) to be a solution of (11b) or (12b).

These equations are a special case of Example 1. The characteristic equations for (11a) are

\[ \frac{dx}{dt} = a, \quad \frac{dy}{dt} = b + \sqrt{b^2 - ac}, \quad \frac{d\xi}{dt} = 0 \]  
(13)

Therefore, \( \xi \) is constant on each characteristic.

Alternatively, as the total derivative of \( \xi \) along the coordinate line \( \xi(x, y) = \text{constant} \), \( d\xi = 0 \), it follows that

\[ d\xi = \xi_x dx + \xi_y dy = 0, \]

and hence the slope of such curves is given by

\[ \frac{dy}{dx} = -\frac{\xi_x}{\xi_y}. \]

We also have similar result along coordinate line \( \eta(x, y) = \text{constant} \), i.e.

\[ \frac{dy}{dx} = -\frac{\eta_x}{\eta_y}. \]

Using these results, equation (8) can be written as

\[ a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 \]

The left hand side is called the characteristic polynomial of the PDE (1) and its zero’s are given by

\[ \frac{dy}{dx} = \frac{b + \sqrt{b^2 - ac}}{a} = \lambda_1(x, y), \]

(14a)

and,

\[ \frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a} = \lambda_2(x, y). \]

(14b)

The characteristics are solutions of these equations.

The function \( \xi \) is constant on the characteristic determined by (14a) and the function \( \eta \) is constant on the characteristic determined by (14b).
Equations (14a) and (14b) are two ordinary differential equations, known as the characteristic equations of the PDE (1). The variables $\xi(x, y)$, and $\eta(x, y)$ are determined by the respective solutions of (14a) and (14b). They are ordinary differential equations for families of curves in the $xy$-plane along which $\xi = \text{constant}$ and $\eta = \text{constant}$. Clearly, these families of curves depend on the coefficients $a, b,$ and $c$ in the original PDE (1).

Integration of equation (14a) leads to the family of curvilinear coordinates $\xi(x, y) = c_1$ while the integration of (14b) gives another family of curvilinear coordinates $\eta(x, y) = c_2$, where $c_1$ and $c_2$ are arbitrary constants of integration. These two families of curvilinear coordinates $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$ are called characteristic curves of the hyperbolic equation (1) or, more simply, the characteristics of the equation. Hence, second-order hyperbolic equations have two families of characteristic curves. The fact that $\delta(L) > 0$, means that the characteristics are real curves in $xy$-plane.

If the coefficients $a, b,$ and $c$ are constants, it is easy to integrate equations (14a) and (14b) to obtain the expressions for change of variables formulas for reducing a hyperbolic PDE to the canonical form. Thus, integration of (14a) and (14b) produces

$$y = \frac{b + \sqrt{b^2 - ac}}{a} x + c_1 \quad \text{and,} \quad y = \frac{b - \sqrt{b^2 - ac}}{a} x + c_2$$

(15a)

or,

$$y - \frac{b + \sqrt{b^2 - ac}}{a} x = c_1 \quad \text{and,} \quad y - \frac{b - \sqrt{b^2 - ac}}{a} x = c_2$$

(15b)

Thus, when the coefficients $a, b,$ and $c$ are constants, the two families of characteristic curves associated with PDE reduce to two distinct families of parallel straight lines. Since the families of curves $\xi = \text{constant}$ and $\eta = \text{constant}$ are the characteristic curves, the change of variables are given by the following equations:

$$\xi = y - \frac{b + \sqrt{b^2 - ac}}{a} x = y - \lambda_1 x$$

(16)

$$\eta = y - \frac{b - \sqrt{b^2 - ac}}{a} x = y - \lambda_2 x$$

(17)

The first canonical form of the hyperbolic equation is:

$$w_{\xi\eta} = \psi(\xi, \eta, w, w_{\xi}, w_{\eta})$$

(18)

where $\psi = \frac{\varphi}{2B}$ and $B$ is calculated from (5) as

$$B(\xi, \eta) = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y$$

$$= a \left(\frac{b^2 - (b^2 - ac)}{a^2}\right) + b \left(\frac{-b}{a} - \frac{b}{a}\right) + c$$
The transformation \( \xi = \xi(x, y) \) and \( \eta = \eta(x, y) \) can be regarded as a mapping from the \( xy \)-plane to the \( \xi \eta \)-plane, and the curves along which \( \xi \) and \( \eta \) are constant in the \( xy \)-plane become coordinates lines in the \( \xi \eta \)-plane. Since these are precisely the characteristic curves, we conclude that when a hyperbolic PDE is in canonical form, coordinate lines are characteristic curves for the PDE. In other words, characteristic curves of a hyperbolic PDE are those curves to which the PDE must be referred as coordinate curves in order that it takes on canonical form.

We now determine the Jacobian of transformation defined by (16) and (17). We have

\[
J = \begin{vmatrix}
-\lambda_1 & 1 \\
-\lambda_2 & 1 \\
\end{vmatrix} = \lambda_2 - \lambda_1
\]

We know that \( \lambda_2 = \lambda_1 \) only if \( b^2 - ac = 0 \). However, for an hyperbolic PDE, \( b^2 - ac \neq 0 \).

Hence Jacobian is nonsingular for the given transformation. A consequence of \( \lambda_2 \neq \lambda_1 \) is that at no point can the particular curves from each family share a common tangent line.

It is easy to show that the hyperbolic PDE has a second canonical form. The following linear change of variables

\[
\alpha = \xi + \eta, \quad \beta = \xi - \eta
\]

converts (18) into

\[
w_{\alpha\alpha} - w_{\beta\beta} = \psi(\alpha, \beta, w, w_\alpha, w_\beta) \tag{20}
\]

which is the second canonical form of the hyperbolic equations.

**Definition 1.** The solutions of (14a) and (14b) are called the two families of the characteristics (or characteristic projections) of the equation \( L[u] = g \).

**Example 1.** Show that the one-dimensional wave equation

\[
u_{tt} - c^2 u_{xx} = 0 \quad \text{i.e.} \quad \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{21}
\]

is hyperbolic, find an equivalent canonical form, and then obtain the general solution.

**Soln.** To interpret the results for (1) that involve the independent variables \( x \) and \( y \) in terms of the wave equation \( u_{tt} - c^2 u_{xx} = 0 \), where the independent variables are \( t \) and \( x \), it will be necessary to replace \( x \) and \( y \) in (1) and (3) by \( t \) and \( x \). It follows that the wave equation is a constant coefficient equation with
\[ a = 1, \quad b = 0, \quad c = -c^2 \]  \hspace{1cm} (22)

We calculate the discriminant, \( \delta(L) = c^2 > 0 \), and therefore the PDE is hyperbolic.

The zero’s of the characteristic polynomial are given by
\[ \lambda_1 = \frac{b+\sqrt{b^2-ac}}{a} = c \quad \text{and} \quad \lambda_2 = \frac{b-\sqrt{b^2-ac}}{a} = -c \]  \hspace{1cm} (23)

Therefore, from the characteristic equations (14a) and (14b) we have
\[ \frac{dx}{dt} = \frac{b+\sqrt{b^2-ac}}{a} = \lambda_1 = c, \quad \text{and} \quad \frac{dx}{dt} = \frac{b-\sqrt{b^2-ac}}{a} = \lambda_2 = -c, \]  \hspace{1cm} (24)

Integrating the above two ODEs, we obtain the characteristics of the wave equation as
\[ x = ct + k_1, \quad \text{and} \quad x = -ct + k_2 \]  \hspace{1cm} (25)

where \( k_1 \) and \( k_2 \) are the constants of integration. We see that the two families of characteristics for the wave equation are given by \( x - ct = \text{constant} \) and \( x + ct = \text{constant} \). It follows, then, that the transformation
\[ \xi = x - ct, \quad \eta = x + ct \]  \hspace{1cm} (26)

reduces the wave equation to canonical form. We have,
\[ A = 0, \quad C = 0, \quad B = -\frac{2\delta}{a} = -2c^2 \]  \hspace{1cm} (27)

So in terms of characteristic variables, the wave equation reduces to the following canonical form
\[ w_{\xi\eta} = 0 \]  \hspace{1cm} (28)

For the wave equation the characteristics are found to be straight lines with negative and positive slopes as shown in the figure below. The characteristics form a natural set of coordinates for the hyperbolic equation.

The canonical forms are simple because they can be solved directly by integrating twice.

For example, integrating (28) with respect to \( \xi \) gives
where the ‘constant of integration’ \( h \) is an arbitrary function of \( \eta \). Next, integrating with respect to \( \eta \) we obtain

\[
\begin{align*}
w(\xi, \eta) &= \int h(\eta) \, d\eta + f(\xi) = f(\xi) + g(\eta)
\end{align*}
\]  

(30)

where \( f \) and \( g \) are arbitrary twice differentiable functions and \( g \) is just the integral of the arbitrary function \( h \). The form of the general solutions of the wave equation in terms of its original variable \( x \) and \( t \) are then given by

\[
\begin{align*}
u &= f(x - ct) + g(x + ct)
\end{align*}
\]  

(31)

Note that \( f \) is constant on “wavefronts” \( x = ct + \xi \) that travel towards right, whereas \( g \) is constant on wavefronts \( x = -ct + \eta \) that travel towards left. Thus, any general solution can be expressed as the sum of two waves, one travelling to the right with constant velocity \( c \) and the other travelling to the left with the same velocity \( c \). This is one of the few cases where the general solution of a PDE can be found.

We also note that hyperbolic PDE has an alternate canonical form with the following linear change of variables \( \alpha = \xi + \eta \) and \( \beta = \xi - \eta \), given by

\[
w_{\alpha\alpha} - w_{\beta\beta} = 0
\]  

(32)

**Example 2.** Consider the Tricomi equation:

\[
u_{xx} + x \nu_{yy} = 0, \quad x < 0.
\]  

(33)

Find a mapping \( q = q(x, y), r = r(x, y) \) that transforms the equation in to its canonical form, and present the equation in this coordinate system.

**Soln.** It follows that the Tricomi equation is a variable coefficient equation with

\[
a = 1, \quad b = 0, \quad c = x, \quad (x < 0)
\]  

(34)

We calculate the discriminant, \( \delta(L) = -x > 0 \), and therefore the PDE is hyperbolic.

The zeroes of the characteristic polynomial are given by

\[
\lambda_1 = \frac{b + \sqrt{b^2 - 4ac}}{a} = \sqrt{-x} \quad \text{and} \quad \lambda_2 = \frac{b - \sqrt{b^2 - 4ac}}{a} = -\sqrt{-x}
\]  

(35)

Therefore, from the characteristic equations (14a) and (14b) we have

\[
\frac{dy}{dx} = \frac{b + \sqrt{b^2 - 4ac}}{a} = \lambda_1 = \sqrt{-x}, \quad \text{and} \quad \frac{dy}{dx} = \frac{b - \sqrt{b^2 - 4ac}}{a} = \lambda_2 = -\sqrt{-x},
\]  

(36)

Integrating the above two ODEs, we obtain the characteristics of the wave equation as

\[
y = -\frac{2}{3} (x)^{3/2} + k_1, \quad \text{and} \quad y = \frac{2}{3} (x)^{3/2} + k_2
\]  

(37)
where \( k_1 \) and \( k_2 \) are the constants of integration. We see that the two families of characteristics for the wave equation are given by \( \frac{3}{2} y + (-x)^{3/2} = \text{constant} \) and \( \frac{3}{2} y - (-x)^{3/2} = \text{constant} \). It follows, then, that the transformation

\[
q = \frac{3}{2} y + (-x)^{3/2}, \quad r = \frac{3}{2} y - (-x)^{3/2}
\]

reduces the Tricomi equation to canonical form.

Thus, the new independent variables are

\[
q(x, y) = \frac{3}{2} y + (-x)^{3/2}, \quad r(x, y) = \frac{3}{2} y - (-x)^{3/2}.
\]

Clearly,

\[
q_x = -r_x = -\frac{3}{2} (-x)^{1/2}, \quad q_y = r_y = \frac{3}{2}.
\]

Define \((q, r) = u(x, y)\). By the chain rule

\[
\begin{align*}
 u_x &= -\frac{3}{2} (-x)^{1/2} v_q + \frac{3}{2} (-x)^{1/2} v_r, \quad u_y = \frac{3}{2} v_q + \frac{3}{2} v_r, \\
 u_{xx} &= -\frac{9}{4} x v_{qq} - \frac{9}{4} x v_{rr} + 2 \frac{9}{4} x v_{qr} + \frac{3}{4} (-x)^{-1/2} (v_q - v_r), \\
 u_{yy} &= \frac{9}{4} (v_{qq} + v_{rr} + 2 v_{qr}).
\end{align*}
\]

Substituting these expressions into the Tricomi equation we obtain

\[
u_{xx} + x u_{yy} = -9(q - r)^2 \left[ v_{qr} - \frac{v_q - v_r}{6(q - r)} \right] = 0
\]

i.e.

\[
v_{qr} = \frac{v_q - v_r}{6(q - r)} = \psi(q, r, v, v_q, v_r) \quad (\because q \neq r)
\]

**Canonical form of parabolic equations**

**Theorem 2.** Suppose that (1) is parabolic in a domain \( D \). There exists a coordinate system \((\xi, \eta)\) in which the equation has the canonical form

\[
w_{\xi\xi} + \ell_1[w] = G(\xi, \eta).
\]

or,

\[
w_{\xi\xi} = \psi(\xi, \eta, w, w_\xi, w_\eta)
\]

where \( w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)) \). \( \ell_1 \) or \( \psi \) is a first-order linear differential operator, and \( G \) is a function which depends on (1).

**Proof.** Since \( b^2 - ac = 0 \), we may assume that \( a(x, y) \neq 0 \) for all \((x, y) \in D\). We need to find two functions \( \xi = \xi(x, y), \eta = \eta(x, y) \) such that \( B(\xi, \eta) = C(\xi, \eta) = 0 \) for all \((x, y) \in D\). It is enough
to make \( C = 0 \), since the parabolicity of the equation will then imply that \( B = 0 \). Therefore, we need to find a function \( \eta \) that is a solution of the equation

\[
C(\xi, \eta) = a\eta_x^2 + 2b\eta_x \eta_y + c\eta_y^2 = \frac{1}{a} (a\eta_x + b \eta_y)^2 = 0. \tag{44}
\]

Dividing the above equation by \( \eta_y^2 \) we have

\[
a \left( \frac{\eta_x}{\eta_y} \right)^2 + 2b \left( \frac{\eta_x}{\eta_y} \right) + c = 0 \tag{45}
\]

From this it follows that \( \eta \) is a solution of the first-order linear equation

\[
a \eta_x + b \eta_y = 0 \tag{46}
\]

Hence, the solution \( \eta \) is constant on each characteristic, i.e., on a curve that is a solution of the equation

\[
\frac{dy}{dx} = \frac{b}{a} \tag{47}
\]

Alternatively, as the total derivative of \( \eta \) along the coordinate line \( \eta(x,y) = \text{constant} \), \( d\eta = 0 \), it follows that

\[
d\eta = \eta_x dx + \eta_y dy = 0,
\]

and hence the slope of such curves is given by

\[
\frac{dy}{dx} = -\frac{\eta_x}{\eta_y}.
\]

Using this result, equation (45) can be written as

\[
a \left( \frac{dy}{dx} \right)^2 + 2b \left( \frac{dy}{dx} \right) + c = 0
\]

The left hand side is called the characteristic polynomial of the PDE (1). Since \( b^2 - ac = 0 \) in this case, it has only one zero given by

\[
\frac{dy}{dx} = \frac{b}{a} = \lambda(x,y) \tag{47a}
\]

The characteristics are solutions of this equation.

The function \( \eta \) is constant on the characteristic determined by (47).

Hence we see that for a parabolic PDE there is only one family of real characteristic curves. The required variables \( \eta \) is determined by the ordinary differential equation (47), known as the characteristic equations of the PDE (1). This is an ordinary differential equation for families of curves in the \( xy \)-plane along which \( \eta = \text{constant} \).

Now, the only constraint on the second independent variable \( \xi \), is that the Jacobian of the transformation should not vanish in \( D \), and we may take any such function \( \xi \). Note that a parabolic equation admits only one family of characteristics while for hyperbolic equations we have two families.
We may also determine the second transformation variable $\xi$ by setting $B(\xi, \eta) = 0$ in (5) so that
\[
a\xi_x \eta_x + b(\xi_x \eta_y + \xi_y \eta_x) + c\xi_y \eta_y = 0 ,
\]
or,
\[
a\xi_x \frac{\eta_x}{\eta_y} + b \left( \xi_x + \xi_y \frac{\eta_x}{\eta_y} \right) + c\xi_y = 0 ,
\]
or,
\[
a\xi_x \left( -\frac{b}{a} \right) + b \left[ \xi_x + \xi_y \left( -\frac{b}{a} \right) \right] + c\xi_y = 0 ,
\]
or,
\[
-b\xi_x + b\xi_x + \xi_y \left( -\frac{b^2}{a} \right) + c\xi_y = 0 ,
\]
or,
\[
\xi_y \left( b^2 - ac \right) = 0 .
\]
Since $b^2 - ac = 0$ for a parabolic PDE, $\xi_y$ could be an arbitrary function of $(x, y)$ and consequently the transformation variable $\xi$ can be chosen arbitrarily, as long as the change of coordinates formulas define a non-degenerate transformation.

If the coefficients $a, b,$ and $c$ are constants, it is easy to integrate equation (47 or 47a) to obtain the expressions for change of variable formulas for reducing a parabolic PDE to the canonical form. Thus, integration of (47) produces
\[
y = \frac{b}{a} x + c_1
\]
(49a)
\[
y - \frac{b}{a} x = c_1
\]
(49b)
Since the families of curves $\eta =$ constant are the characteristic curves, the change of variables are given by the following equations:
\[
\eta = y - \frac{b}{a} x
\]
(50)
\[
\xi = x
\]
(51)
where we have set $\xi = x$. The Jacobian of this transformation is
\[
J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{b}{a} & 1 \end{vmatrix} = 1 \neq 0
\]
Now, we have from (5) and (48)
\[
B = a\xi_x \eta_x + b(\xi_x \eta_y + \xi_y \eta_x) + c\xi_y \eta_y = a \left( -\frac{b}{a} \right) + b = 0
\]
In these new coordinate variables given by (50) and (51), equation (4) reduces to following canonical form:
\[
w_{\xi \xi} = \psi(\xi, \eta, w, w_{\xi}, w_{\eta})
\]
(52)
where $\psi = \frac{w}{A}$. As the choice of $\xi$ is arbitrary, the form taken by $\psi$ will depend on the choice of $\xi$. We have from (5)
\[
A(\xi, \eta) = a\xi_x^2 + 2b\xi_x \eta_x + c\eta_y^2 = a + 2b \left( -\frac{b}{a} \right) + c = \frac{a^2 - b^2}{a}
\]
(53)
Equation (4) may also assume the form
\[
w_{\eta \eta} = \psi(\xi, \eta, w, w_{\xi}, w_{\eta})
\]
(54)
if we choose $A = 0$ instead of $C = 0$. 

Example 3. Show that the one-dimensional heat equation
\[ \alpha u_{xx} = u_t \quad \text{i.e.} \quad \alpha \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad (55) \]
is parabolic, choose the appropriate characteristic variables, and write the equation in equivalent canonical form.

Soln. It follows that the heat equation is a constant coefficient equation with
\[ a = \alpha, \quad b = 0, \quad c = 0 \quad (56) \]
We calculate the discriminant, \( \delta(L) = 0 \), and therefore the PDE is parabolic.

The single zero of the characteristic polynomial is given by
\[ \lambda = \frac{b}{a} = 0 \quad (57) \]
Therefore, from the characteristic equation (47a) we have
\[ \frac{dt}{dx} = \frac{b}{a} = 0, \quad (58) \]
Integrating the above ODE, we obtain the characteristics of the wave equation as
\[ t = k, \quad (59) \]
where \( k \) is the constant of integration. We see that only one family of characteristics for the heat equation is given by \( t = \text{constant} \).

Since the families of curves \( \eta = \text{constant} \) are the characteristic curves, the change of variables are given by the following equations:
\[ \eta = t \quad (60) \]
\[ \xi = x \quad (61) \]
where we have set \( \xi = x \).

This shows that the given PDE is already expressed in canonical form and thus no change of variable is needed to simplify the structure. Further, we have from (3)
\[ u_t = w_\xi \xi_t + w_\eta \eta_t = w_\eta \quad (62) \]
and by (53),
\[ A = \frac{a^2 - b^2}{a} = \alpha \quad (63) \]
It follows that the canonical form of the heat equation is given by
\[ w_{\xi\xi} = \frac{1}{\alpha} w_\eta \quad (64) \]

Example 4. Prove that the equation
\[ x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} + xu_x + yu_y = 0 \quad (65) \]
is parabolic and find its canonical form; find the general solution on the half-plane \( x > 0 \).
Soln. We identify $a = x^2$, $2b = -2xy$, $c = y^2$; therefore,

$$b^2 - ac = x^2y^2 - x^2y^2 = 0,$$

and the equation is parabolic.

The equation for the characteristics is

$$\frac{dy}{dx} = -\frac{y}{x},$$

and the solution is $xy = \text{constant}$.

Therefore, we define $\eta(x, y) = xy$.

The second variable can be simply chosen as $\xi(x, y) = x$.

Let $v(\xi, \eta) = u(x, y)$.

Substituting the new coordinates $\xi$ and $\eta$ into (65), we obtain

$$x^2(y^2v_{\eta\eta} + 2yxv_{\xi\eta} + v_{\xi\xi}) - 2xy(v_{\eta} + xyy_{\eta} + x\xi_{\eta}) + x^2v_{\eta\eta} + xyv_{\eta} + x\xi_{\eta} + xyv_{\eta} = 0. \tag{71}$$

Thus,

$$\xi^2v_{\xi\xi} + \xi v_{\xi} = 0,$$

or

$$v_{\xi\xi} + (1/\xi)v_{\xi} = 0, \tag{72}$$

and this is the desired canonical form.

Setting $w = v_{\xi}$,

we arrive at the first-order ODE

$$w_{\xi} + (1/\xi)w = 0. \tag{74}$$

The solution is

$$\ln w = -\ln \xi + f(\eta), \tag{75}$$

or

$$w = f(\eta)/\xi. \tag{76}$$

Hence, $v$ satisfies

$$v = \int v_{\xi} d\xi = \int wd\xi = \int \frac{f(\eta)}{\xi} d\xi = f(\eta) \ln \xi + g(\eta), \tag{77}$$

Therefore, the general solution $u(x, y)$ of (65) is

$$u(x, y) = f(xy) \ln x + g(xy), \tag{78}$$

where $f, g \in C^2(R)$ are arbitrary real functions.

**Canonical form of elliptic equations**

The computation of a canonical coordinate system for the elliptic case is somewhat more subtle than in the hyperbolic case or in the parabolic case. Nevertheless, under the additional assumption that the coefficients of the principal part of the equation are real analytic functions, the procedure for determining the canonical transformation is quite similar to the one for the hyperbolic case.
Definition 2. Let $D$ be a planar domain. A function $f : D \rightarrow \mathbb{R}$ is said to be real analytic in $D$ if for each point $(x_0, y_0) \in D$, we have a convergent power series expansion
\[
f(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{k} a_{i,j-k}(x-x_0)^i(y-y_0)^{k-j},
\] valid in some neighbourhood $N$ of $(x_0, y_0)$.

Theorem 3. Suppose that $(1)$ is elliptic in a planar domain $D$. Assume further that the coefficients $a, b, c$ are real analytic functions in $D$. Then there exists a coordinate system $(\xi, \eta)$ in which the equation has the canonical form
\[
w_{\xi \xi} + w_{\eta \eta} + \ell_1[w] = G(\xi, \eta),
\]
or,
\[
w_{\xi \xi} + w_{\eta \eta} = \psi(\xi, \eta, w, w_{\xi}, w_{\eta})
\]
where $\ell_1$ or $\psi$ is a first-order linear differential operator, and $G$ is a function which depends on $(1)$.

Proof. Without loss of generality we may assume that $a(x, y) \neq 0$ for all $(x, y) \in D$.

We are looking for two functions $\xi = \xi(x, y), \eta = \eta(x, y)$ that satisfy the equations
\[
A(\xi, \eta) = a \xi^2 + 2b \xi \eta + c \eta^2 = C(\xi, \eta) = a_1 \xi + 2b_1 \eta + c_1 \eta^2 = 0,
\]
\[
B(\xi, \eta) = a \xi \eta + b(\xi \eta + \xi \eta) + c \xi \eta = 0
\]
This is a system of two nonlinear first-order equations. The main difficulty in the elliptic case is that (81) and (82) are coupled. In order to decouple these equations, we shall use the complex plane and the analyticitiy assumption. We may write the system (81)–(82) in the following form:
\[
a(\xi^2 - \eta^2) + 2b(\xi \eta - \eta \xi) + c(\xi^2 - \eta^2) = 0
\]
\[
a \xi i \eta + b(\xi i \eta + \xi i \eta) + c \xi i \eta = 0
\]
where $i = \sqrt{-1}$. Define the complex function $\phi = \xi + i \eta$. The system (83)–(84) is equivalent to the complex valued equation [obtained by multiplying (84) by 2 and then adding to (83)]
\[
a \phi_x^2 + 2b \phi_x \phi_y + c \phi_y^2 = 0,
\]
or,
\[
a \left(\frac{\phi_x}{\phi_y}\right)^2 + 2b \left(\frac{\phi_x}{\phi_y}\right) + c = 0
\]
Surprisingly, we have arrived at the same equation as in the hyperbolic case [see eqn. (7a,b)]. But in the elliptic case the equation does not admit any real solution, or, in other words, elliptic equations do not have characteristics. As in the hyperbolic case, we factor out the above quadratic PDE, and obtain two linear equations, but now these are complex valued differential equations (where $x, y$ are complex variables!). The nontrivial question of the existence and uniqueness of solutions immediately arises. Fortunately, it is known that if the coefficients of these first-order linear equations are real analytic then it is possible to solve them using the same procedure as in the real case. Moreover, the solutions of the two equations are complex conjugates.

So, we need to solve the equations
\[ a\phi_x + (b \pm i\sqrt{ac-b^2})\phi_y = 0 \]  
or,  
\[ \frac{\phi_x}{\phi_y} = -\frac{b \pm i\sqrt{ac-b^2}}{a} \]  
(86a)

These two roots are complex conjugates (say \(\varphi, \psi\)) and are given by
\[ \frac{\varphi_x}{\varphi_y} = -\frac{b - i\sqrt{ac-b^2}}{a} \]
(86b)
\[ \frac{\psi_x}{\psi_y} = -\frac{b + i\sqrt{ac-b^2}}{a} \]
(86c)

where \(\varphi(x,y) = \xi(x,y) + i\eta(x,y)\) and \(\psi(x,y) = \xi(x,y) - i\eta(x,y)\).

As before, the solutions \(\varphi, \psi\) are constant on the “characteristics” (which are defined on the complex plane):
\[ \frac{dy}{dx} = \frac{b \pm i\sqrt{ac-b^2}}{a} \]
(87)

Alternatively, as the total derivative of \(\varphi\) along the coordinate line \(\varphi = \text{constant}, d\varphi = 0\), it follows that
\[ d\varphi = \varphi_x dx + \varphi_y dy = 0 \]
and hence, the slope of such curves is given by
\[ \frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} \]

We also have a similar result along coordinate line \(= \text{constant}\), i.e.
\[ \frac{dy}{dx} = -\frac{\psi_x}{\psi_y} \]

From the above discussion it follow again, that
\[ \frac{dy}{dx} = \frac{b - i\sqrt{ac-b^2}}{a} \]
(87a)
and,  
\[ \frac{dy}{dx} = \frac{b + i\sqrt{ac-b^2}}{a} \]
(87b)

Equations (87a) and (87b) are called the characteristic equation of the PDE (1). Clearly, the solutions of these differential equations are necessarily complex-valued and as a consequence there are no real characteristic exist for an elliptic PDE.

The complex variables \(\varphi\) and \(\psi\) are determined by the respective solutions of the two ordinary differential equations (87a) and (87b). Integration of equation (87a) leads to the family of curvilinear coordinates \(\varphi(x,y) = c_1\) while the integration of (87b) gives another family of curvilinear coordinates \(\psi(x,y) = c_2\), where \(c_1\) and \(c_2\) are complex constants of integration.

Since \(\varphi\) and \(\psi\) are complex function the characteristic curves of the elliptic equation (1) are not real.

As in the hyperbolic case, the equation in the new coordinates system has the form
\[ 4\nu_{\varphi\psi} + \cdots = 0 \]
(88)

This is still not the elliptic canonical form with real coefficients. We return to our real variables \(\xi\) and \(\eta\) using the linear transformation
\( \xi = Re \phi, \quad \eta = Im \phi . \)

Since \( \xi \) and \( \eta \) are solutions of the system (81)–(82), it follows that in the variables \( \xi \) and \( \eta \) the equation has the canonical form.

**Example 5.** Consider the Tricomi equation:

\[ u_{xx} + xu_{yy} = 0, \quad x > 0. \]  
(89)

Find a canonical transformation \( q = q(x, y), \quad r = r(x, y) \) and the corresponding canonical form.

The differential equations for the “characteristics” are

\[ \frac{dv}{dx} = \pm \sqrt{-x}, \]  
(90)

and their solutions are

\[ \frac{3}{2} y \pm i(x)^{3/2} = \text{constant}. \]  
(91)

Therefore, the canonical variables are

\[ q(x, y) = \frac{3}{2} y \quad \text{and} \quad r(x, y) = -(x)^{3/2} \]  
(92)

Clearly,

\[ q_x = 0, \quad q_y = \frac{3}{2}; \quad r_x = -\frac{3}{2} (x)^{1/2}, \quad r_y = 0. \]  
(93)

Set \( v(q, r) = u(x, y). \)  
(94)

Hence,

\[ u_x = -\frac{3}{2} (x)^{1/2} v_r, \quad u_y = \frac{3}{2} v_q, \]  
(95)

\[ u_{xx} = \frac{9}{4} x v_{rr} - \frac{3}{4} (x)^{-1/2} v_r, \quad u_{yy} = \frac{9}{4} v_{qq}. \]

Substituting these into the Tricomi equation we obtain the canonical form

\[ \frac{1}{x} u_{xx} + u_{yy} = \frac{9}{4} \left( v_{qq} + v_{rr} + \frac{1}{3r} v_r \right) = 0 . \]  
(96)

or,

\[ v_{qq} + v_{rr} = -\frac{1}{3r} v_r . \]  
(97)

**Example 6.** Show that the equation

\[ u_{xx} + x^2 u_{yy} = 0 \]  
(98)

is elliptic everywhere except on the coordinate axis \( x = 0 \), find the characteristic variables and hence write the equation in canonical form.

**Soln.** The given equation is of the form (1) where

\[ a = 1, \quad b = 0, \quad c = x^2 \]  
(99)

The discriminant, \( \delta = b^2 - ac = -x^2 < 0 \) for \( x \neq 0 \), and therefore the PDE is elliptic.

The roots of the characteristic polynomial are given by

\[ \lambda_1 = \frac{b + i\sqrt{ac - b^2}}{a} = -ix, \quad \text{and} \quad \lambda_2 = \frac{b - i\sqrt{ac - b^2}}{a} = ix \]  
(100)
Therefore, from the characteristic equations (14a) and (14b), we have

\[ \frac{dy}{dx} = -ix, \quad \frac{dx}{dy} = ix \]  \hspace{1cm} (101)

Integrating the above two ODEs we obtain the characteristics in the complex plain as

\[ y = -i \frac{x^2}{2} + c_1, \quad y = i \frac{x^2}{2} + c_2 \]  \hspace{1cm} (102)

where \( c_1 \) and \( c_2 \) are the complex constants. We see that the two families of complex characteristics for the elliptic equation are given by

\[ y + i \frac{x^2}{2} = \text{constant} \quad \text{and} \quad y - i \frac{x^2}{2} = \text{constant}. \]  \hspace{1cm} (103)

It follows, then, that the transformation

\[ \alpha = y + i \frac{x^2}{2}, \quad \beta = y - i \frac{x^2}{2}. \]  \hspace{1cm} (104)

The real and imaginary parts of \( \alpha \) and \( \beta \) give the required transformation variables \( \xi \) and \( \eta \).

Thus, we have

\[ \xi = \frac{\alpha + \beta}{2} = y, \quad \eta = \frac{\alpha - \beta}{2i} = \frac{x^2}{2}. \]  \hspace{1cm} (105)

With these choice of coordinate variables, equation (4) reduces to following canonical form.

From the relations (3), we have

\[ u_{xx} = w_{\xi \xi} \xi_x^2 + 2w_{\xi \eta} \xi_x \eta_x + w_{\eta \eta} \eta_x^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx} \]
\[ = x^2 w_{\eta \eta} + w_\eta \]  \hspace{1cm} (106)

\[ u_{yy} = w_{\xi \xi} \xi_y^2 + 2w_{\xi \eta} \xi_y \eta_y + w_{\eta \eta} \eta_y^2 + w_\xi \xi_{yy} + w_\eta \eta_{yy} \]
\[ = w_{\xi \xi} \]  \hspace{1cm} (107)

Substituting these relations in the given PDE and noting that \( x^2 = 2 \eta \), we obtain

\[ w_{\xi \xi} + w_{\eta \eta} = -\frac{1}{2\eta} w_\eta \]  \hspace{1cm} (108)

This is the canonical form of the given elliptic PDE.

Therefore, the PDE

\[ u_{xx} + x^2 u_{yy} = 0 \]

in rectangular coordinate system \((x,y)\) has been transformed to PDE

\[ w_{\xi \xi} + w_{\eta \eta} = -\frac{1}{2\eta} w_\eta \]

in curvilinear coordinate system \( (\xi, \eta) \). Here \( \xi = \text{const.} \) lines represents a family of straight lines parallel to \( x \) axis and \( \eta = \text{const.} \) lines represents family of parabolas.