2016

MATHEMATICS — HONOURS

Fifth Paper

(Module – X)

Full Marks – 50

The figures in the margin indicate full marks

Candidates are required to give their answers in their own words as far as practicable

Group – A
(Marks – 20)
Section – I

Answer *any one* question

1. (a) Let V and W be two vector spaces over a field F and $T : V \to W$ be a linear mapping. If $\text{Ker}\,T = \{\theta\}$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V, prove that $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a basis of Im T. 5

(b) The matrix of a linear mapping $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with respect to the ordered basis $((0,1,1),(1,0,1),(1,1,0))$ of $\mathbb{R}^3$ is given by

$$\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}$$

Find T. Find also the matrix of T relative to the ordered basis $((2,1,1),(1,2,1),(1,1,2))$. 2+3

2. (a) Let V be a vector space of dimension n over a field F. Prove that V is isomorphic of F^n. 5

(b) Let a linear mapping $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be defined by $T(x_1, x_2, x_3, x_4) = (3x_1 - 2x_2 - x_3 - 4x_4,\ x_1 + x_2 - 2x_3 - 3x_4)$, $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Find rank T, nullity T and a basis of Ker T. 5

Section – II

Answer *any one* question

3. (a) If N is a normal sub group of a group $(G, \circ)$ and G/N is the set of all left cosets of N in G, prove that G/N is a group under the binary operation $*$ given by

$$(aN) * (bN) = (a \circ b)N \text{ for all } a, b \in G.$$ 5

(b) Let $f : (\mathbb{Z}, +) \to (Q - \{0\}, \cdot)$ be defined by

$$f(m) = \begin{cases} 1, & \text{if } m \text{ is even} \\ -1, & \text{if } m \text{ is odd} \end{cases}$$

[*Turn Over*]

show that

(i) f is a homomorphism.

(ii) Determine Ker f and prove that $\mathbb{Z}/\text{Ker } f$ is isomorphic to $f(\mathbb{Z})$.

(Z and Q are the set of integers and rational numbers respectively). 2+1+2

4. (a) Let $\phi:(G,\circ)\rightarrow(G',*)$ be an isomorphism. Prove that G' is cyclic if and only if G is cyclic. 5

(b) Let $a, b \in \mathbb{R}$ and a mapping $T_{ab} : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $T_{ab}(x) = ax+b$, $x \in \mathbb{R}$. Let $G = \{T_{ab} : a \neq 0\}$. Assume that $(G, *)$ is a group where $*$ is the composition of mappings. If $H = \{T_{ab} : a = 1\}$, prove that $(H, *)$ is a normal subgroup of G. 5

Group – B

(Marks – 15)

Answer ***any three*** questions

5. (a) If A_i is a covariant vector, prove that $\frac{\partial A_i}{\partial x^j}$ is not a tensor.

(b) If a^{ij} are components of a symmetric tensor prove that

$$a^{jk}\,[ij, k] = \frac{1}{2} a^{jk} \frac{\partial g_{jk}}{\partial x^i}$$

where g_{ik} are components of the metric tensor of the Riemannian space. 2+3

6. If a_{ij} and a^{ji} are reciprocal symmetric tensors of the second order, prove that

(a) $a^{ij}\frac{\partial a_{ij}}{\partial x^k} + a_{ij}\frac{\partial a^{ij}}{\partial x^k} = 0$

(b) $\frac{\partial \log a}{\partial x^k} = a^{ij}\frac{\partial a_{ij}}{\partial x^k} = -a_{ij}\frac{\partial a^{ij}}{\partial x^k}$,

where $a = |a_{ij}|$. 5

7. If $a_{ij}(\neq 0)$ are the components of a covariant tensor of order 2 such that $b\,a_{ij} + c\,a_{ji} = 0$ where b and c are non-zero scalars, show that either $b=c$ and a_{ij} is skew-symmetric or $b=-c$ and a_{ij} is symmetric. 5

8. In E_3, line element in spherical co-ordinates x^1, x^2, x^3 is given by $(ds)^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1 \sin x^2)^2 (dx^3)^2$. Find g^{ij}. 5

9. Find the covariant derivative A_{ij}, k of a (0, 2) type tensor A_{ij} and prove that A_{ij}, k is symmetric in i and j if A_{ij} is symmetric. 4+1

Answer either **Group – C** or **Group – D**

Group – C

(Marks – 15)

Answer ***any one*** question

10. (a) If $L\{F(t)\} = f(p)$ and $G(t) = F(t-a),\ t > a$
$\qquad\qquad = 0 \qquad ,\ t < a$

then show that $L(G(t)) = e^{-ap} L(F(t))$ and hence find the Laplace transform of F(t) where

$$F(t) = \begin{cases} \sin\left(t - \frac{\pi}{3}\right), & t \ge \pi/3 \\ 0 & , t < \pi/3 \end{cases}$$ 2+3

(b) Using Laplace transform, solve

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4e^{2t},$$

given $y(0) = -3$ and $y'(0) = 5$. 5

(c) Solve the equation $2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1 - x^2)y = 0$ in series near the ordinary point $x = 0$. 5

11. (a) Apply convolution theorem to evaluate 5

$$L^{-1}\left[\frac{s}{\left(s^2 + a^2\right)^2}\right].$$

(b) Find inverse Laplace transform of 5

$$\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}.$$

(c) Solve the equation 5

$$x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0$$

in series near the ordinary point $x = 0$.

Group – D

(Marks – 15)

Answer ***any three*** questions

12. Prove that the number of odd degree vertices of a graph G is always even. Also prove that the number of edges of a complete graph with n vertices is $\frac{n(n-1)}{2}$. 3+2

13. (a) If a graph G is Eulerian then prove that degree of every vertex of G is even.

(b) Justify why every tree is a bipartite graph. 3+2

14. (a) Distinguish between walk and circuit in a graph.

(b) Does there exist a tree G with 10 vertices such that the total degree of G is 24 ? Justify your answer. 3+2

15. Prove that the utility graph $K_{3,3}$ is non-planar. 5

16. Prove that a connected graph with n vertices has at least (n–1) edges. 5