2015

MATHEMATICS — HONOURS

Third Paper

(Module – VI)

Full Marks – 50

The figures in the margin indicate full marks

Candidates are required to give their answers in their own words as far as practicable

$\mathbb{R}$ denotes the set of real numbers

Group – A

(Analysis – II)

(Marks – 15)

Answer ***any three*** questions

1. (a) Let $\{a_n\}_n$ be a monotone decreasing sequence of positive real numbers. Prove that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converge or diverge together. 3

Or

Prove that a conditionally convergent series of real numbers is expressible as difference of two divergent series. 3

(b) Correct or justify the following statement :

If a series of real numbers $\sum_{n=1}^{\infty} x_n$ is convergent and the real sequence $\{y_n\}_n$ is bounded, then $\sum_{n=1}^{\infty} x_n y_n$ is convergent. 2

2. (a) Test the convergence of ***any one*** of the following series : 3×1

(i) $\frac{1}{1^{1/2}}+\frac{1}{2^{1/2}}+\frac{1}{3^{1/2}}+\frac{1}{1^{1/3}}+\frac{1}{2^{1/3}}+\frac{1}{3^{1/3}}+\frac{1}{1^{1/4}}+\frac{1}{2^{1/4}}+\frac{1}{3^{1/4}}+\ldots\ldots$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{1}{n}$.

(b) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of real numbers. Can you conclude that $\sum_{n=1}^{\infty} n^{1/n} a_n$ is convergent ? Justify your answer. 2

3. (a) A function $f:[a,b]\to \mathbb{R}$ is continuous on $[a, b]$ and $f''(x)$ exists for all $x\in(a, b)$. If $a < c < b$ and $f(a) = f(b) = 0$, prove that there exists a point $\xi\in(a, b)$ such that

$$f(c)=\frac{1}{2}(c-a)(c-b)f''(\xi)$$ 3

(b) Let $f:[a,b]\to \mathbb{R}$ be differentiable on $[a, b]$ and f' be bounded on $[a, b]$. Prove that f satisfies Lipschitz condition on $[a, b]$. 2

4. (a) Show that $\dfrac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}=\cot\theta$ where $0<\alpha<\theta<\beta<\dfrac{\pi}{2}$. 3

(b) Let $g:[0, 2]\to \mathbb{R}$ be defined by

$$g(x)=\begin{cases} 3 & \text{if } 0\le x\le 1 \\ -1 & \text{if } 1<x\le 2 \end{cases}$$

Does there exist a function F on $[0, 2]$ such that $F'(x)=g(x)\ \forall\ x\in[0, 2]$? Justify your answer. 2

5. (a) The perimeter of an isosceles triangle is 2s. What must its sides be so that the volume of the solid generated by revolving the triangle about the base is the greatest possible ? 3

Or

Determine a, b, c if

$$\lim_{t\to 0}\frac{ae^t-b\cos t+ce^{-t}}{t\sin t}=2$$ 3

(b) Prove that $x = 0$ is a saddle point of the function $f(x)=x^3$, $x\in \mathbb{R}$. 2

Group – B

(Differential Equation – I)

(Marks – 35)

Answer *any five* questions

6. Reduce the differential equation

$$x^2p^2+y(2x+y)p+y^2=0 \quad \left(p\equiv\frac{dy}{dx}\right)$$

to Clairaut's form by the substitution $y = u$, $xy = v$ and then solve it. Obtain also the singular solution, if any. 7

7. State the theorem on existence and uniqueness of solution of the initial value problem

$$\frac{dy}{dx} = f(x,y), \ y(x_0) = y_0$$

Using this theorem, prove that the initial value problem

$$\frac{dy}{dx} = xy + y^3, \ y(0) = 0$$

has a solution for $|x| \le \frac{1}{2}$. 2+5

8. (a) Construct a second order ordinary differential equation whose solution space has $(x^2, \sin x)$ as a basis vector. 3

(b) Solve : $x^3D^3y + 2x^2D^2y - xDy + y = x\log_e x$. $\left(D \equiv \frac{d}{dx}\right)$ 4

9. (a) Determine the orthogonal trajectory of $r = a(2\sin\theta + \cos\theta)$, '$a$' being a parameter. 3

(b) Prove that the ordinary differential equation

$$p_0(x)y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0, \quad x \in D \subset \mathbb{R}$$

is exact if and only if $p_0''(x) - p_1'(x) + p_2(x) = 0 \quad \forall x \in D$. 4

10. (a) Using the method of undetermined coefficients, solve :

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = e^{-2x}(1 + \cos x)$$ 5

(b) Factorise the differential expression

$$(x+2)y''(x) - (2x+5)y'(x) + 2y(x).$$ 2

11. Solve $(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6(x^2+1)^2$ by the method of variation of parameters. 7

12. Solve the initial value problem : 7

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t$$

$$\frac{dx}{dt} - \frac{dy}{dt} - 2x = 4\cos t,$$

$$(x(0), y(0)) = (0, 0)$$

13. (a) Determine the eigenvalues and the eigenfunctions of the boundary value problem : 5

$$\frac{d^2y}{dx^2} - 4\lambda\frac{dy}{dx} + 4\lambda^2 y = 0,$$
$$y(0) + y'(0) = 0,$$
$$y(1) + y'(1) = 0$$

(b) Reduce the ordinary differential equation

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(a^2 + \frac{2}{x^2}\right)y = 0$$

to normal form. 2

14. (a) Form the partial differential equation by eliminating the arbitrary constants a, b, c from the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$ 3

(b) Solve : $px + qy = z - a\sqrt{x^2 + y^2 + z^2}$

where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$. 4