2015

MATHEMATICS — HONOURS

Fourth Paper

(Module – VII)

Full Marks – 50

The figures in the margin indicate full marks

Candidates are required to give their answers in their own words as far as practicable

[IR denotes the set of all real numbers]

Group – A

(Marks – 30)

Answer *any six* questions

1. Correct or justify the following :

(a) The set $A = \{(x, y) : x, y \text{ both are rationals}\}$ is neither open nor closed set in $\mathbb{R}^2$. 2

(b) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the functions $f(x) = F(x, b)$ and $g(y) = F(a, y)$ are continuous at $x = a$ and $y = b$ respectively. Then F is continuous at (a, b). 3

2. Let
$$f(x, y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \cos\frac{1}{y}, & x \neq 0, y \neq 0 \\ x^2 \sin\frac{1}{x}, & x \neq 0 \\ y^2 \cos\frac{1}{y}, & y \neq 0 \\ 0, & (x, y) = (0, 0) \end{cases}$$

Prove that f_x, f_y exist at (0, 0) but none is continuous. Examine differentiability of f at (0, 0). 5

3. Let (a, b) be an interior point of domain of a function f of two variables. If f_x and f_y be both differentiable at (a, b), prove that $f_{xy}(a, b) = f_{yx}(a, b)$. Cite an example to show that the condition is not necessary. 3+2

4. If $u = F\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$ be a differentiable function, show that

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$$ 5

[*Turn Over*]

5. State Euler's theorem on homogeneous function of two variables. If u, v are two polynomials in x, y that are homogeneous of degree n, prove that

$$udv - vdu = \frac{1}{n}\frac{\partial(u, v)}{\partial(x, y)}(xdy - ydx), \text{ where } \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0.$$ 1+4

6. If $u = \dfrac{x}{\sqrt{1-x^2-y^2-z^2}}$, $v = \dfrac{y}{\sqrt{1-x^2-y^2-z^2}}$, $w = \dfrac{z}{\sqrt{1-x^2-y^2-z^2}}$

evaluate $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$. 5

7. Transform the equation $\dfrac{\partial^2 z}{\partial x^2} + 2xy^2\dfrac{\partial z}{\partial x} + 2(y - y^3)\dfrac{\partial z}{\partial y} + x^2.y^2 z = 0$ by the substitution $x = uv$, $y = \dfrac{1}{v}$.

Hence show that z is the same function of u and v as of x and y. 4+1

8. Show that $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$ are functionally related. Also find the relation among them. 5

9. (a) Show that $y^2 - x^2 y - 2x^5 = 0$ determines y uniquely as a function of x near the point (1, – 1) and find $\dfrac{dy}{dx}$ at (1, – 1). 2+1

(b) Find the saddle points for the function $f(x, y) = xy(1 - x^2 - y^2)$ on the square $0 \le x \le 1$, $0 \le y \le 1$. 2

10. Show that there exists θ with 0 < θ < 1 such that

$$\sin x.\sin y = xy - \frac{1}{6}\left[(x^3 + 3xy^2)\cos\theta x.\sin\theta y + (y^3 + 3x^2 y)\sin\theta x.\cos\theta y\right].$$ 5

11. Find the triangle of maximum area having the fixed perimeter 2p. 5

Group – B

(Marks – 20)

Answer ***any four*** questions

12. Prove that the pedal of the rectangular hyperbola $x^2 - y^2 = a^2$ with respect to the centre is $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. 5

13. Find the asymptotes of the curve $x = \dfrac{t^2}{1+t^3}$, $y = \dfrac{t^2+2}{1+t}$. 5

14. Find the equation of the circle of curvature of the curve

$$x = e^{-2t}\cos 2t,\ \ y = e^{-2t}\sin 2t \ \text{ at } t = 0.$$ 5

15. Find the envelope of the family of straight lines drawn at right angles to the radii vectors of the cardioide $r = a(1+\cos\theta)$ through their extremities. 5

16. Find the range of values of x for which $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is concave upwards or downwards. Find also its point of inflexion, if any. 2+2+1

17. Find the area of the region bounded by the curve $y = x(x-1)(x-2)$ and the x-axis. 5

18. Find the co-ordinate of centre of gravity of an arc of the astroid $x = a\cos^3 t,\ y = a\sin^3 t$ lying in the first quadrant. 5